Polarisation and spatial curvature

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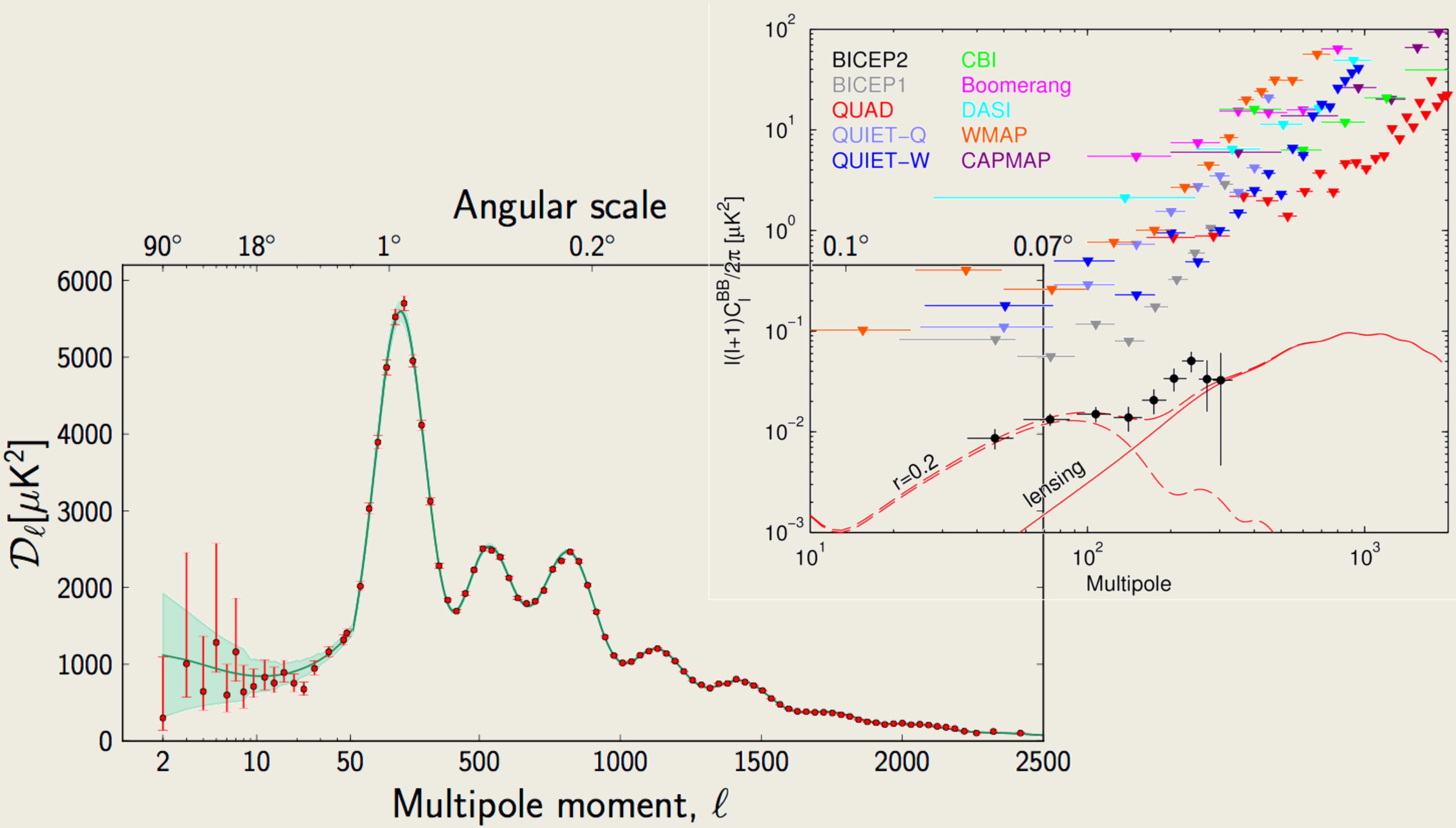
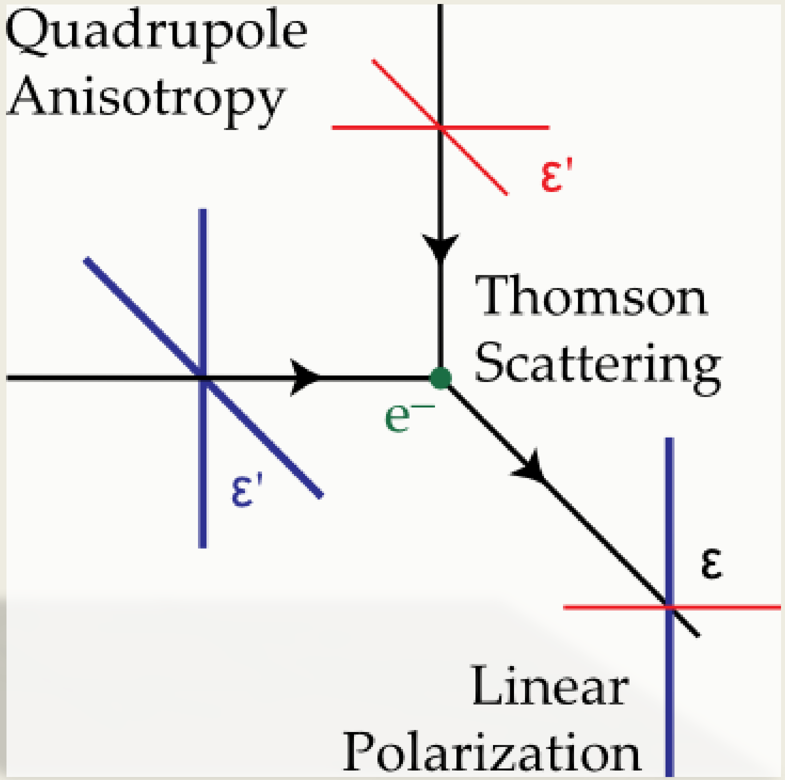
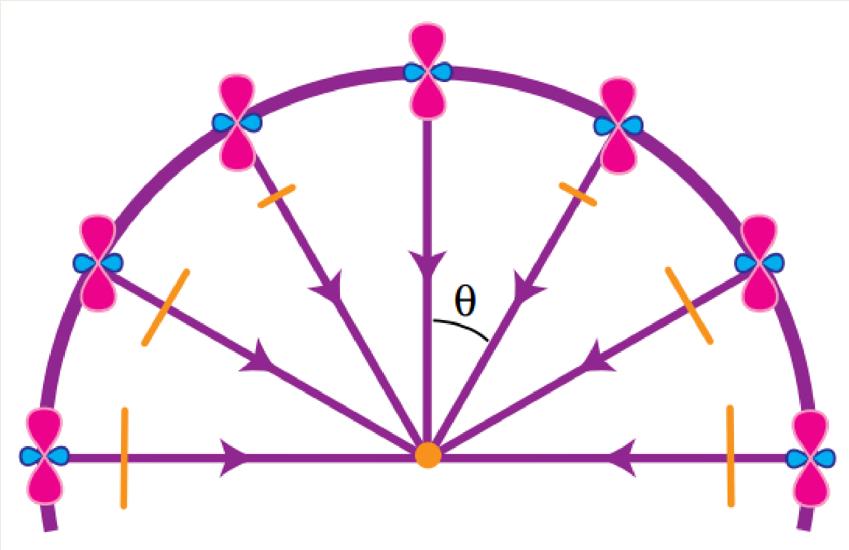
CMB observables



Part1: CMB polarisation





Credits: Hu&White, astro-ph/9706147

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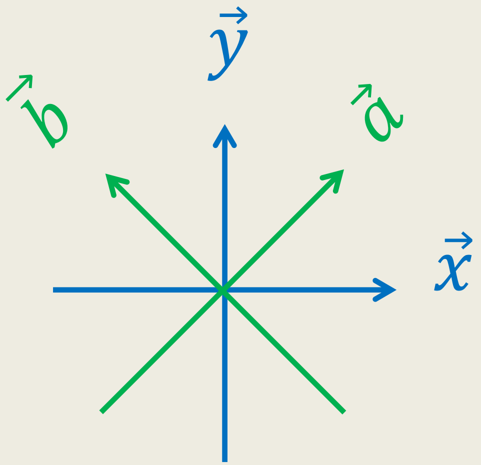
Stokes parameters

- A general radiation field is described by the 4 Stokes parameters:

$$
\begin{aligned}
I &= \langle E_x^2 \rangle + \langle E_y^2 \rangle \\
Q &= \langle E_x^2 \rangle - \langle E_y^2 \rangle \\
U &= \langle E_a^2 \rangle - \langle E_b^2 \rangle \\
V &= -2\mathrm{Im}(\langle E_x E_y^* \rangle)
\end{aligned}
$$



- They form the intensity matrix

$$
\mathcal{J} = \begin{bmatrix} I + Q & U - \mathbb{i}V \\ U + \mathbb{i}V & I - Q \end{bmatrix} = I\mathbf{1} + Q\sigma_3 + U\sigma_1 + V\sigma_2
$$

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Stokes parameters

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$$I = \langle E_x^2 \rangle + \langle E_y^2 \rangle$$
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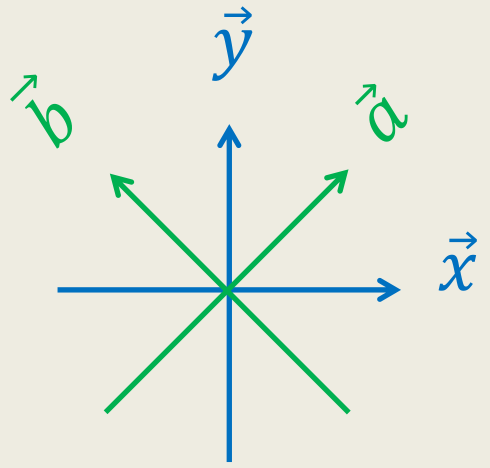

- They form the intensity matrix

$$\mathcal{J} = \begin{bmatrix} I + Q & U \\ U & I - Q \end{bmatrix} = I\mathbf{1} + Q\sigma_3 + U\sigma_1$$

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E and B polarisation $\mathcal{J} = \begin{bmatrix} I+Q & U \\ U & I-Q \end{bmatrix}$

- Rank 2 tensor field $\mathcal{J}$ on the sphere is covariant, not Q and U. (Why is I?)
- But $Q \pm \mathbb{i}U \to e^{\mp 2\mathbb{i}\psi}(Q \pm \mathbb{i}U)$
- Expand in spin ± 2 spherical harmonics ${}_{\pm 2}Y_l^m$:

$$(Q \pm \mathbb{i}U)(\vec{x}, \vec{n}) \sim$$
$$\int d^3k e^{\mathbb{i}\vec{k}\cdot\vec{x}} \sum (-\mathbb{i})^l \left\{ E_l^{(m)} \pm \mathbb{i}B_l^{(m)} \right\} {}_{\pm 2}Y_l^m$$

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Photon Boltzmann equation

- Complicated because of geometry:
  - Simple derivation using ${}_sY_l^m$ harmonics thanks to Hu&White: astro-ph/9702170
  - Use vector $\vec{T} = (\Theta, Q + \mathbb{i}U, Q - \mathbb{i}U)$
- Boltzmann equation:

$$\frac{d\vec{T}}{d\tau} = \frac{\partial \vec{T}}{\partial \tau} + n^i \vec{T}_{|i} = \vec{C}[\vec{T}] + \begin{pmatrix} D_\Theta \\ 0 \\ 0 \end{pmatrix}$$

Photon Boltzmann equation II

$$\frac{d}{d\tau}\begin{pmatrix}\Theta \\ Q+\mathbb{i}U \\ Q-\mathbb{i}U\end{pmatrix} + \dot{\kappa}\begin{pmatrix}\Theta - \hat{n}\cdot\vec{v}_b - \int\frac{d\Omega'}{4\pi}\Theta' \\ Q+\mathbb{i}U \\ Q-\mathbb{i}U\end{pmatrix} - \begin{pmatrix}D_\Theta \\ 0 \\ 0\end{pmatrix} = \frac{\dot{\kappa}}{10}\int d\Omega'$$

$$\sum_{m=-2}^{2}\begin{pmatrix} Y_2^{m\prime} Y_2^m & -\sqrt{\frac{3}{2}}\,{}_2Y_2^{m\prime} Y_2^m & -\sqrt{\frac{3}{2}}\,{}_{-2}Y_2^{m\prime} Y_2^m \\ -\sqrt{6}Y_2^{m\prime}\,{}_2Y_2^m & 3\,{}_2Y_2^{m\prime}\,{}_2Y_2^m & 3\,{}_{-2}Y_2^{m\prime}\,{}_2Y_2^m \\ -\sqrt{6}Y_2^{m\prime}\,{}_{-2}Y_2^m & 3\,{}_2Y_2^{m\prime}\,{}_{-2}Y_2^m & 3\,{}_{-2}Y_2^{m\prime}\,{}_{-2}Y_2^m \end{pmatrix}\begin{pmatrix}\Theta' \\ Q'+\mathbb{i}U' \\ Q'-\mathbb{i}U'\end{pmatrix}$$

- Details not important, but the structure is.

Hu&White: astro-ph/9702170
Hu,White,Seljak,Zaldarriaga: astro-ph/9709066

Photon Boltzmann equation III

$$\frac{d}{d\tau}\begin{pmatrix}\Theta \\ Q \\ \mathbb{i}U\end{pmatrix} + \dot{\kappa}\begin{pmatrix}\Theta - \hat{n}\cdot\vec{v}_b - \int \frac{d\Omega'}{4\pi}\Theta' \\ Q \\ \mathbb{i}U\end{pmatrix} - \begin{pmatrix}D_\Theta \\ 0 \\ 0\end{pmatrix} =$$

$$\frac{\dot{\kappa}}{10}\sum_{m=-2}^{2}\int d\Omega' \begin{pmatrix} Y_2^m\left\{Y_2^{m\prime}\Theta' - \sqrt{\frac{3}{2}}\mathcal{E}'^m Q' - \sqrt{\frac{3}{2}}\mathcal{B}^{m\prime}\mathbb{i}U'\right\} \\ \frac{1}{2}\mathcal{E}^m\left\{-\sqrt{6}Y_2^{m\prime}\Theta' + 3\mathcal{E}'^m Q' + 3\mathcal{B}^{m\prime}\mathbb{i}U'\right\} \\ \frac{1}{2}\mathcal{B}^m\left\{-\sqrt{6}Y_2^{m\prime}\Theta' + 3\mathcal{E}'^m Q' + 3\mathcal{B}^{m\prime}\mathbb{i}U'\right\}\end{pmatrix}$$

where $\mathcal{E}^m \stackrel{\text{def}}{=} {}_2Y_2^m + {}_{-2}Y_2^m$ and $\mathcal{B}^m \stackrel{\text{def}}{=} {}_2Y_2^m + {}_{-2}Y_2^m$.

But since this equation holds separately for each m...

Photon Boltzmann equation IV

...we must have $\mathbb{i}U^{(m)} = \frac{\mathcal{B}^m}{\mathcal{E}^m} Q^{(m)}$!

$$\frac{d}{d\tau}\begin{pmatrix}\Theta^{(m)} \\ Q^{(m)}\end{pmatrix} + \dot{\kappa}\begin{pmatrix}\Theta^{(\mathrm{m})} - \hat{n}\cdot\vec{v}_b^{(m)} - \int\frac{d\Omega'}{4\pi}\Theta' \\ Q^{(m)}\end{pmatrix} - \begin{pmatrix}D_\Theta \\ 0\end{pmatrix} =$$

$$\frac{\dot{\kappa}}{10}\int d\Omega' \begin{pmatrix} Y_2^m\left\{Y_2^{m'}\Theta' - \sqrt{\frac{3}{2}}\left[\mathcal{E}'^m + \frac{\left(\mathcal{B}^{m'}\right)^2}{\mathcal{E}'^m}\right]Q'\right\} \\ -\sqrt{\frac{3}{2}}\mathcal{E}'^m\left\{Y_2^{m'}\Theta' - \sqrt{\frac{3}{2}}\left[\mathcal{E}'^m + \frac{\left(\mathcal{B}^{m'}\right)^2}{\mathcal{E}'^m}\right]Q'\right\}\end{pmatrix}$$

J.Lesgourgues&TT:arXiv:1305.3261

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Relation to E and B

- Line-of-sight solutions needed for efficient implementation
  - CMBfast, Seljak&Zaldarriaga 1996
  - Known for scalars, vectors and tensors in non-flat universes in terms of Θ_l, E_l and B_l (HWSZ 1998)
  - Done if we can relate the multipole moments of ${\Theta_l}^{(m)}$, $E_l^{(m)}$ and $B_l^{(m)}$ to $F_l^{(m)}$ and $G_l^{(m)}$

Tedious but possible...

$$\Theta_l^{(0)} = \frac{2l+1}{4} F_l^{(0)}$$

$$E_l^{(0)} = \frac{2l+1}{4} F_l^{(0)} \sqrt{\frac{(l-2)!}{(l+2)!}} \left(-l(l-1) G_l^{(0)} + \sum_{\substack{k=0, \\ k+l \text{ even}}}^{l-2} 2\mathbb{i}^{l-k} (2k+1) G_k^{(0)} \right)$$

⋮ (For vector modes see paper)

$$\Theta_l^{(2)} = -\frac{1}{4} \sqrt{\frac{(l+2)!}{(l-2)!}} \left(\frac{1}{2l-1} F_{l-2}^{(2)} + \frac{2(2l+1)}{(2l-1)(2l+3)} F_l^{(2)} + \frac{1}{2l+3} F_{l+2}^{(2)} \right)$$

$$E_l^{(2)} = \sqrt{\frac{2l+1}{5}} \left(-(2l-3)\alpha_{l-2}^l G_{l-2}^{(2)} + (2l+1)\alpha_l^l G_l^{(2)} - (2l+5)\alpha_{l+2}^l G_{l+2}^{(2)} \right)$$

$$B_l^{(2)} = \sqrt{\frac{2l+1}{5}} \left((2l-1)\alpha_{l-1}^l G_{l-1}^{(2)} - (2l+3)\alpha_{l+1}^l G_{l+1}^{(2)} \right)$$

The line-of-sight integrals

- We have computed the source, now convolve with certain radial functions:

$$\frac{\Theta_l^{(m)}}{2l+1} = \sum_j \int_0^{\tau_0} d\tau e^{-\kappa} S_j^{(m)} \phi_l^{(jm)}$$

$$\frac{E_l^{(m)}}{2l+1} = -\int_0^{\tau_0} d\tau \dot{\kappa} e^{-\kappa} \sqrt{6} P^{(m)} \epsilon_l^{(m)}$$

$$\frac{B_l^{(m)}}{2l+1} = -\int_0^{\tau_0} d\tau \dot{\kappa} e^{-\kappa} \sqrt{6} P^{(m)} \beta_l^{(m)}$$

Part 2: The radial functions

- The radial functions are linear combinations of Φ_l^{ν}, $\Phi_l^{\nu}{}'$ and $\Phi_l^{\nu}{}''$.
- In the flat limit: $j_l(x)$, ν dependence becomes a rescaling of argument. ($x = k(\tau_0 - \tau)$)
- During MCMC, we must compute them on the fly => Much longer execution time
- However: Flat rescaling approximation!

Hypergeometric Bessel functions

- FLRW metric:

$$ds^2 = a(\tau)^2\left[-d\tau^2 + \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right],$$
$$= a(\tau)^2[-d\tau^2 + d\chi^2 + r^2 d\Omega^2]$$
$$r(\chi) = \begin{cases} \sin\chi & K=1 \\ \chi & K=0 \\ \sinh\chi & K=-1 \end{cases}$$

- Radial part of $\nabla^2 F = -k^2 F$ is equivalent to:

$$u'' = \left[\frac{l(l+1)}{r(\chi)^2} - \nu^2\right]u, \qquad u = r(\chi)\Phi(\chi).$$

What is going on???

- The Boltzmann equation is a partial differential equation :

$$\frac{d\vec{T}}{d\tau} = \frac{\partial \vec{T}}{\partial \tau} + n^i \vec{T}_{|i} = \vec{C}[\vec{T}] + \begin{pmatrix} D_\Theta \\ 0 \\ 0 \end{pmatrix}$$

- We do not like to solve PDEs, only ODEs. So we employ a *spectral method*: we expand $\vec{T}$ in eigenfunctions of the differential operator.

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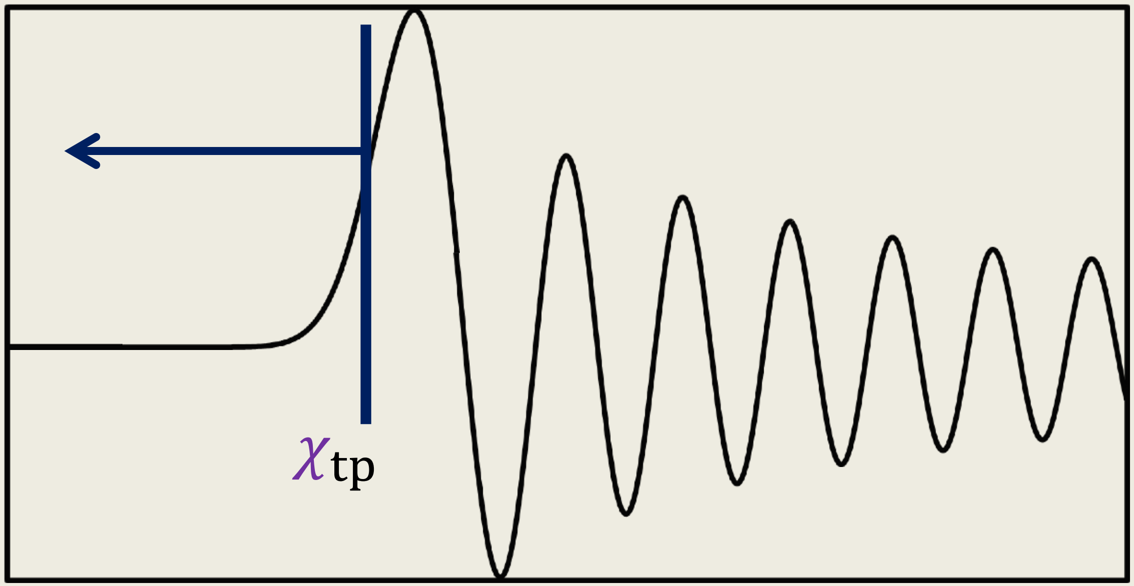
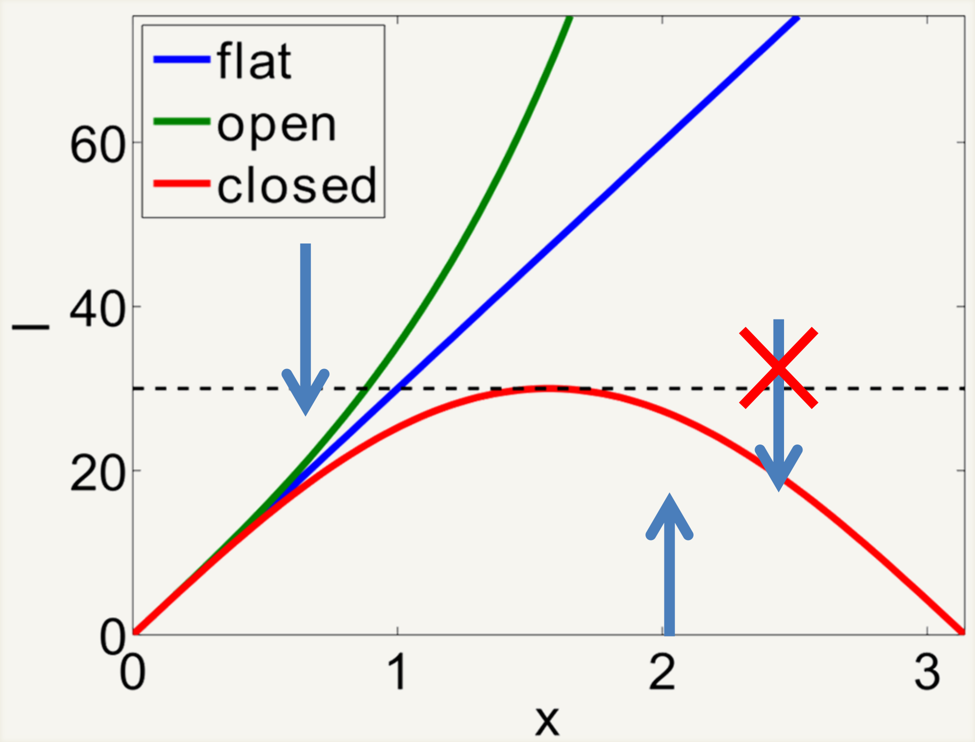
Schrödinger-like equation

- We have a classical turningpoint, separating the dissipative region $0 < \chi < \chi_{\mathrm{tp}}$ from the dispersive region $\chi_{\mathrm{tp}} < \chi < \infty$



Standard recurrence method

- Backwards recurrence in dissipative region, forwards recurrence in dispersive region.



But:
Closed model restricted by $l < \nu$, so no backwards recurrence...

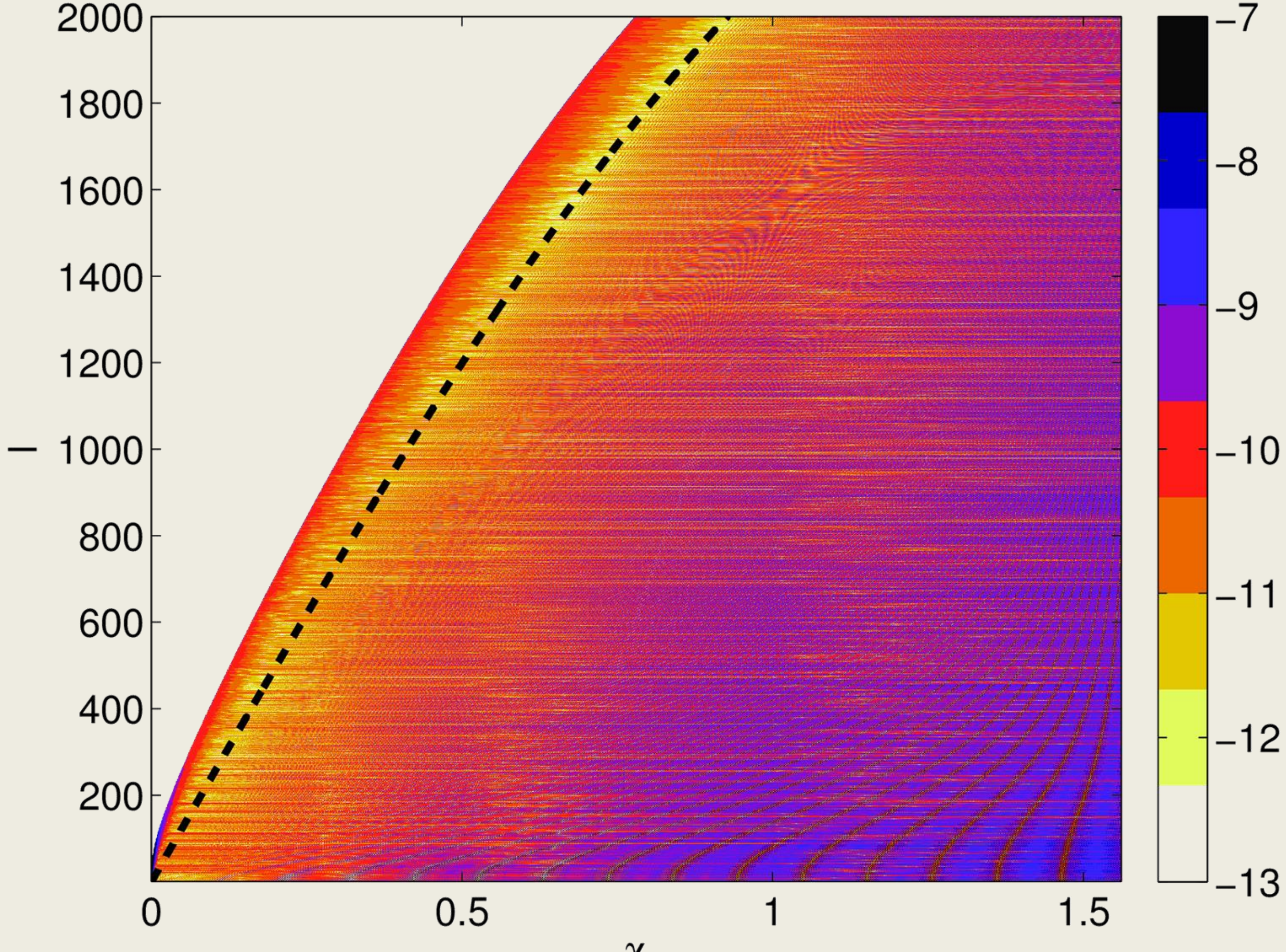
Relation to Gegenbauer polynomials

- We found the following important identity:

$$\Phi_l^\nu(x) = 2^l l! \sqrt{\frac{(\nu-l-1)!}{\nu(\nu+l)!}} \sin^l(x)\, C_{\nu-l-1}^{l+1}(\cos(x))$$

- Consider $l = \nu - 1$, then $C_0^{l+1}(\cos(x)) = 1$, and we have downwards recurrence!

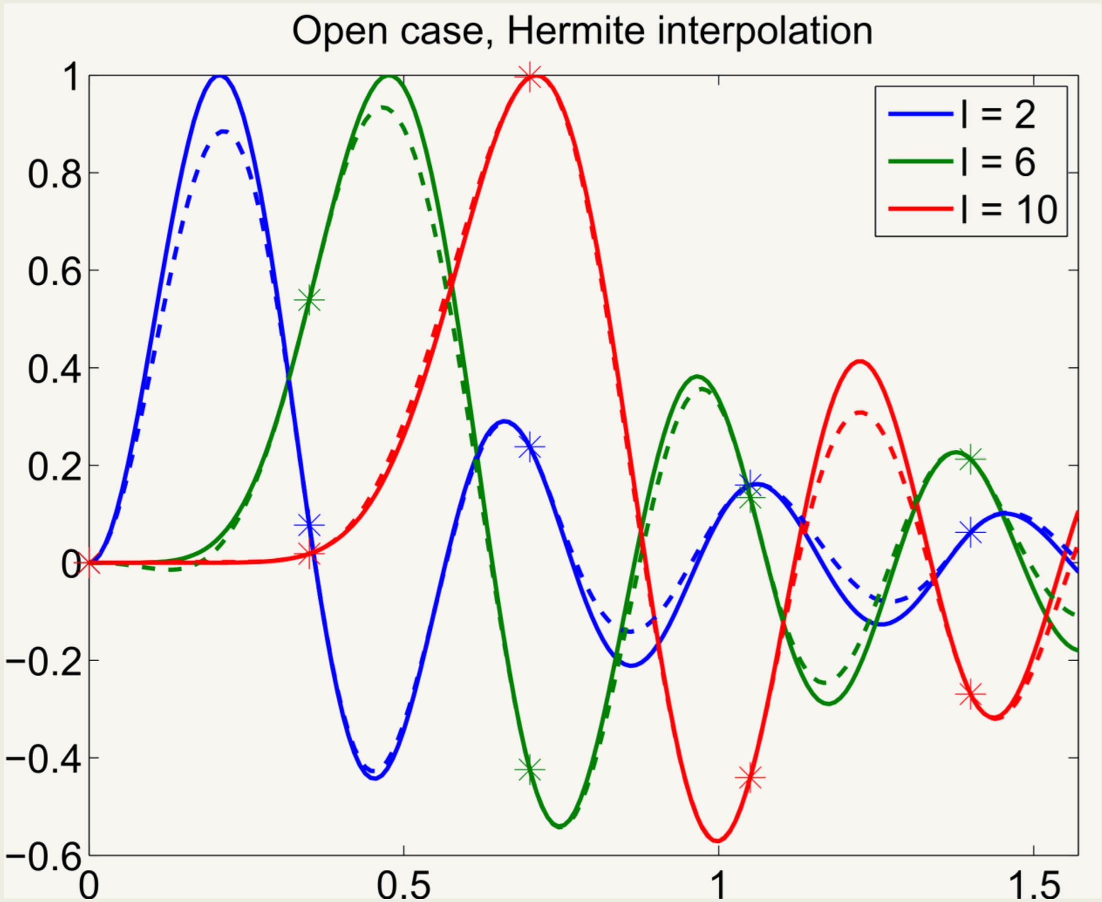
It works!



2000
1800
1600
1400
1200
l
1000
800
600
400
200
0
0.5
1
1.5
χ
−7
−8
−9
−10
−11
−12
−13

Hermite interpolation

- Store Φ and Φ', compute higher order derivatives.
- Use 6 constraints => 5th order polynomial



Note: Only 5 computed points!