# Chapter I — Brief recalls on homogeneous cosmology

## I.1. Introduction

Background metric $ds^2 = dt^2 - a^2(t)\, d\vec{x}^2$ — units $c=1$

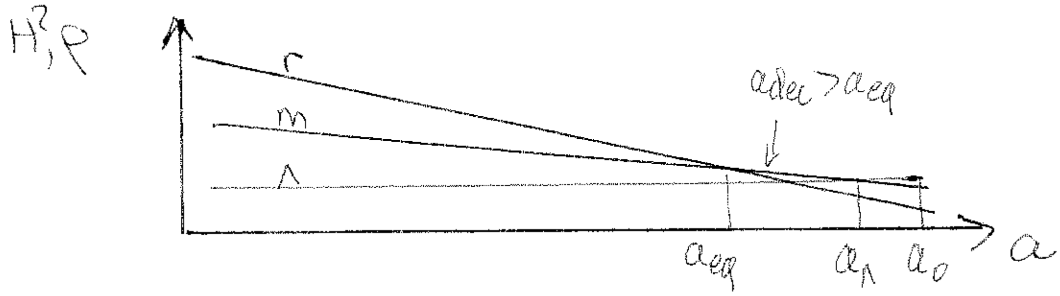
(in all this course: flat universe for simplicity)

Friedmann equation: $H^2 = \frac{8\pi G}{3} \rho_{tot} = \frac{8\pi}{3} M_P^{-2} \rho_{tot}$ — $M_P^{-2} \equiv G$

In flat ΛCDM model, $\rho_{tot}$ = sum of three components:

* matter (cdm + baryons) $\rho_m = \rho_c + \rho_b \propto a^{-3}$
* radiation (photons + neutrinos) $\rho_r = \rho_\gamma + \rho_\nu \propto a^{-4}$
* cosmological constant $\rho_\Lambda = cte$

In fact, since total neutrino mass $M_\nu \neq 0$, $\rho_\nu$ dilutes like $a^{-3}$ at late times.




Cosmological parameters:

* $H_0 = 100\, h\, \text{km.s}^{-1}.\text{Mpc}^{-1} = \frac{h}{3000}\, \text{Mpc}^{-1}$ (if $c=1$)
* Friedmann implies matter budget equation:

$$1 = \frac{8\pi G}{3H^2} \sum \rho_i = \frac{\sum \rho_i}{\rho_c} = \sum \Omega_i$$

with $\rho_c \equiv \frac{3H^2}{8\pi G}$ and $\Omega_i \equiv \frac{\rho_i}{\rho_c}$, $\omega_i \equiv \Omega_i h^2$

Usually, $\Omega_i$ means $\frac{\rho_i^0}{\rho_c^0}$ (otherwise we write $\Omega_i(a)$)

* $\Omega_i \equiv$ relative density, while
* $\omega_i \equiv$ absolute density (in some units) since

$$\rho_i^0 = \rho_c^0 \Omega_i = \frac{3(H_0/h)^2}{8\pi G} \omega_i = 1.8788 \times 10^{-29} \omega_i \ g.cm^{-1}$$

In simplest ΛCDM scenario, which quantities are fixed and which are not?

A) Nearly fixed quantities in ΛCDM

1) Density of radiation

Today $T_{cmb} = 2.726 \ K \equiv T_\gamma^0$

$$\rho_\gamma = \underbrace{\frac{2}{(2\pi)^3}}_{\text{2 d.o.f.}} \int d^3p \ \underbrace{p}_{\text{Energy}} \ \underbrace{\frac{1}{e^{p/T_\gamma} - 1}}_{\text{Bose-Einstein}} = 2 \cdot \frac{\pi^2}{30} T_\gamma^4 \Rightarrow \underline{\omega_\gamma \simeq 2.10^{-5}}$$

Let's add neutrinos. First, if we assume $M_\nu = 0$:

$$\rho_\nu = \underbrace{\frac{6}{(2\pi)^3}}_{\text{6 d.o.f. for } \nu_e, \bar\nu_e, \nu_\mu, \bar\nu_\mu, \nu_\tau, \bar\nu_\tau} \int d^3p \ p \ \underbrace{\frac{1}{e^{p/T_\nu} + 1}}_{\text{Fermi-Dirac}} = 6 \cdot \underbrace{\frac{7}{8}}_{\text{fermion factor}} \cdot \frac{\pi^2}{30} T_\nu^4 = 3 \cdot \frac{7}{8} \cdot \left(\frac{T_\nu}{T_\gamma}\right)^4 \rho_\gamma$$

Electron-positron annihilation in instantaneous ν decoupling limit $\Rightarrow T_\nu / T_\gamma = (4/11)^{1/3}$

So $\rho_r = \left[1 + 3 \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{4/3}\right] \rho_\gamma = 1.68 \rho_\gamma$

$\omega_r \simeq 4.10^{-5}$

Taking into account the fact that $M_\nu \neq 0$, previous result not true because for $T_\nu \ll M_\nu$:

$$\rho_\nu \simeq M_\nu n_\nu = M_\nu \frac{6}{(2\pi)^3} \int d^3p \frac{1}{e^{p/T_\nu}+1} = M_\nu \, 6 \frac{\zeta(3)}{\pi^2} T_\nu^3$$

number density

$$\Rightarrow \omega_\nu \simeq \frac{M_\nu}{94 \text{ eV}} \quad \text{with } \frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}$$

However, previous result $\rho_r = 1.68 \rho_\gamma$ remains useful for calculation refering to times such that $T_\nu \gg M_\nu$.
For instance: at R/M equality, neutrinos still non-relativistic for realistic values of $M_\nu$: $M_\nu \lesssim 0.7$eV.
So $a_{eq}$ and $z_{eq}$ only depend on matter density:

$$\rho_m = \rho_r \Leftrightarrow \Omega_m \rho_c^0 \left(\frac{a_0}{a}\right)^3 = \Omega_r \rho_c^0 \left(\frac{a_0}{a}\right)^4$$

So $\frac{a_0}{a_{eq}} \simeq z_{eq} \simeq \frac{\Omega_m}{\Omega_r} = \frac{\omega_m}{\omega_r} = \frac{\omega_m}{4.10^{-5}}$

For $\omega_m \sim 0.1$, this gives $z_{eq} \sim 3000$.

## 2 Time of decoupling

At first order, $a_{dec}$ dictated by balance of reaction

$$e^- + p \longleftrightarrow H + \gamma$$

Balance parametrized by ionisation fraction $X_e \equiv \frac{n_p}{n_p + n_H} = \frac{n_{e^-}}{n_{e^-} + \gamma}$

As long as chemical equilibrium holds, $X_e$ given by Saha equation.

Then, $X_e$ depends on:

→ binding energy of H versus temperature:

$$X_e \propto e^{-\mathcal{E}_0/T_\gamma} \text{ with } \mathcal{E}_0 = 13.6\,\text{eV}$$

→ electron mass versus temperature ($m_e \simeq 0.5\,\text{MeV}$)

→ balance between baryon and photon number density, parametrized by $\Omega_b$ (roughly $n_B \sim 10^{-10} n_\gamma$)

When $X_e \ll 1$, photons decouple. In fact, reaction goes out of chemical equilibrium and freezes out soon after $X_e \ll 1$. However Saha equation gives good first-order prediction of $z_{dec}$ (redshift of photon decoupling). Since only free parameter in Saha is $\Omega_b$, $z_{dec}$ is just a function of $\Omega_b$. However this dependence is very small (because of quick variation of exponential $e^{-\mathcal{E}_0/T_\gamma}$) and can be neglected, leading to nearly fixed value

$$z_{dec} \sim 1100 < z_{eq}$$

## B) Quantities to be measured in ΛCDM

Physically, three quantities remain to be measured accurately:

* $a_\Lambda$ (equality $\Lambda$ / M)
* $a_{eq}$ (equality M / R)
* relative abundance of baryons versus CDM

These 3 effects can be parametrized e.g. by basis $\{h, \Omega_b, \Omega_{cdm}\}$ or $\{\omega_b, \omega_m, \Omega_\Lambda\}$ or etc...

then $\Omega_\Lambda = 1 - \Omega_b - \Omega_{cdm}$ then $h = \sqrt{\frac{\omega_m}{1-\Omega_\Lambda}}$

The goal of studying and observing cosmological perturbations is to:

* measure these 3 parameters
* open window on extra physics and parameters (e.g. related to primordial spectrum, star formation, etc.)
* constrain deviations from minimal ΛCDM (and better test this scenario!)

## I.2. Hubble radius, horizon(s) and observable universe

here exceptionally we restore $c \neq 1$

Hubble radius $R_H \equiv c/H = 3000\, h^{-1}$ Mpc represents "radius of curvature of $(t, x_i)$ sections of space-time" with $i = 1, 2$ or $3$.

→ if $a = cte$, $ds^2 =$ Minkowski : curvature of space-time exists only when $\dot{a} \neq 0$ (and $R_H \neq \infty$)

→ Newtonian interpretation of redshift based on Doppler effect: $z = \frac{v}{c} = \frac{Hr}{c} = \frac{r}{R_H}$ ($v = Hr$: "Hubble flow")

But $v \leq c \Rightarrow r \leq \frac{c}{H} = R_H$

This interpretation makes sense only for $r \leq R_H$

→ more generally : for all phenomena on scales $\lambda \ll R_H$ we can neglect the description of the Universe expansion based on general relativity

→ since $R_H$ = ONLY quantity with a dimension in flat FLRW model, many quantities are related to $R_H$ : age of Universe, causal horizon, radius of observable Universe

## a) Age

$$H = \frac{\dot{a}}{a} = \frac{da}{a\,dt} \qquad \text{with } \frac{a_0}{a} = 1+z \Rightarrow -\frac{a_0\,da}{a^2} = dz$$

$$\text{So } t = \int_{BB}^{t_0} dt = \int_{BB}^{t_0} \frac{da}{aH} = -\int_{BB}^{0} \frac{1}{aH} \frac{a^2\,dz}{a_0} = \int_{0}^{BB} \frac{dz}{(1+z)H}$$

Big Bang

$$\text{Friedmann} \Rightarrow H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda) \text{ and } H_0^2 = \frac{8\pi G}{3}\rho_c^0$$

$$\text{So } (H/H_0)^2 = \Omega_m (1+z)^3 + \Omega_r (1+z)^4 + \Omega_\Lambda$$

$$\text{So } t = \int_0^\infty \frac{dz}{(1+z) H_0} \left[\Omega_m (1+z)^3 + \Omega_r (1+z)^4 + \Omega_\Lambda\right]^{-1/2}$$

$$= H_0^{-1} f(\Omega_m, \underbrace{\Omega_r}_{\text{weak dependence}}, \underbrace{\Omega_\Lambda}_{\simeq 1-\Omega_m}) = \begin{pmatrix}\text{number}\\ \text{of order}\\ \text{one}\end{pmatrix} \times R_H$$

## b) Conformal age

$$a\,d\tau \equiv dt \quad \leftarrow \text{conformal time}$$

$$ds^2 = a^2 (d\tau^2 - d\vec{x}^2)$$

$$\tau = \int_{BB}^{\tau_0} d\tau = \int_{BB}^{t_0} \frac{dt}{a} = \int_{BB}^{0} a_0^{-1} \frac{dz}{H}$$

The age $t$ did not depend on $a_0$, but the conformal age does; only $(a_0 \tau)$ is physical. This is consistent with the meaning of conformal time:
"conformal time = measure of time corresponding to the comoving distance travelled by a free photon from that time until now".
Indeed, on photon geodesics, $dt = a d\tau \Leftrightarrow \underset{\substack{\uparrow \\ \text{conformal time}}}{d\tau} = \underset{\substack{\uparrow \\ \text{comoving distance}}}{dr}$

Since $\tau$ is in fact like a comoving distance (for $c=1$) $(a_0 \tau)$ is a physical distance.
So conformal age = "comoving distance travelled by photon from Big Bang until now, converted in lenght scales of today."

c) Causal horizon

For photon (or information travelling at speed of light), $dr = \frac{dt}{a}$
Causal horizon = comoving distance travelled by photon between time 1 and time 2, expressed in physical lenght scale of time 2:

$$\underbrace{d(t_1,t_2)}_{\text{horizon}} = a_2 \underbrace{\int_{t_1}^{t_2} \frac{dt}{a}}_{\text{comoving horizon } d_c(t_1,t_2)}$$

Definition of horizon makes reference to time 1 because we usually need the horizon "relevant for a physical process starting at time 1" (e.g: acoustic oscillations during RD)

radiation domination

Sometimes, implicitely assumed that causal

$$\text{horizon} = \lim_{t_1 \to \text{BigBang}} d(t_1, t_2)$$

E.g. during RD: Friedmann $\Rightarrow a \propto t^{1/2}$, so

$$d(t_1,t_2) = 2\sqrt{t_2}\left(\sqrt{t_2}-\sqrt{t_1}\right) \xrightarrow[t_2 \gg t_1]{} 2t_2$$

while $R_H = 2t_2$

So, in the limit in which $t_1$ is well before $t_2$,

$$d(t_1,t_2) \simeq R_H(t_2)$$

For process starting during RD, Hubble radius plays role of causal horizon.

## d) Radius of observable Universe

Observation based on light: maximum distance probed is $d_{obs} = a_0 \int_{t_{dec}}^{t_0} \frac{dt}{a} = d(t_{dec}, t_0)$

$\gamma$ decoupling today

Observation based on neutrinos: same with $t_{dec}$ = neutrino decoupling

We could devise on observations based on gravitational waves, etc..

In practise, all definitions nearly equal to each other and to the result of:

$$d_{obs} = a_0 \int_0^{t_0} \frac{dt}{a} \quad \text{with } a \propto t^{1/2} \text{ extrapolated till } t=0$$

We can estimate $d_{obs}$ neglecting RD and $\Lambda$D.

$$\underset{\text{radiation domination}}{\text{RD}} \qquad \underset{\Lambda \text{ domination.}}{\Lambda\text{D}}$$

During MD: $a \propto t^{2/3}$, so:

$$\left.\begin{aligned} d_{obs} &\simeq t_0^{2/3} \int_0^{t_0} \frac{dt}{t^{2/3}} = 3t_0 \\ R_H &= 3t_0/2 \end{aligned}\right\} d_{obs} = 2 R_H$$

Restoring $\Lambda$D and RD, on gets:

$$d_{obs} = f(\underbrace{\Omega_r}_{\text{very weak dependence}}, \Omega_m, \underbrace{\Omega_\Lambda}_{1-\Omega_m}) R_H = \begin{pmatrix}\text{number} \\ \text{of order} \\ \text{one}\end{pmatrix} \times R_H$$

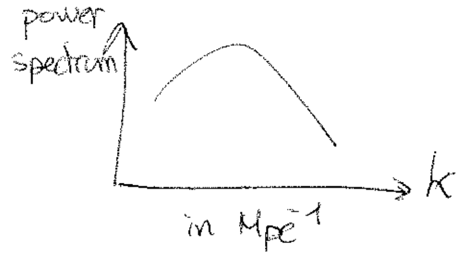
Hence, on very general basis, $R_H$ is a good approximation for the radius of the observable universe

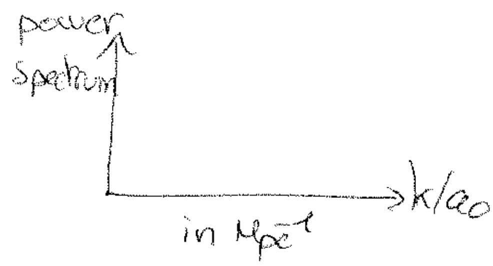
## II.3. Fourier expansion in FLRW universe:

When nothing happens, comoving observer remain at fixed comoving coordinate. Hence, makes sense to define Fourier expansion with respect to comoving coordinate $\vec{x}$ (not with respect to $a(t)\vec{x}$ !!!)

So physical wavelenght of perturbation is a time-dependent quantity: $\lambda(t) = a(t)\frac{2\pi}{k}$

→ remark 1: $k$ cannot be expressed in physical units because only $k/a$ is physical ($k/a$ = inverse of distance). However, observers often report $k$ in units of inverse distance, e.g. $\mathrm{Mpc}^{-1}$. In fact they mean: "$k$ in inverse distances of today", ie they don't mean $k$ but $k/a_0$ (equivalent if we choose $a_0=1$).

Hence 

is in fact 


→ remark 2: time of Hubble crossing for $k$ given by

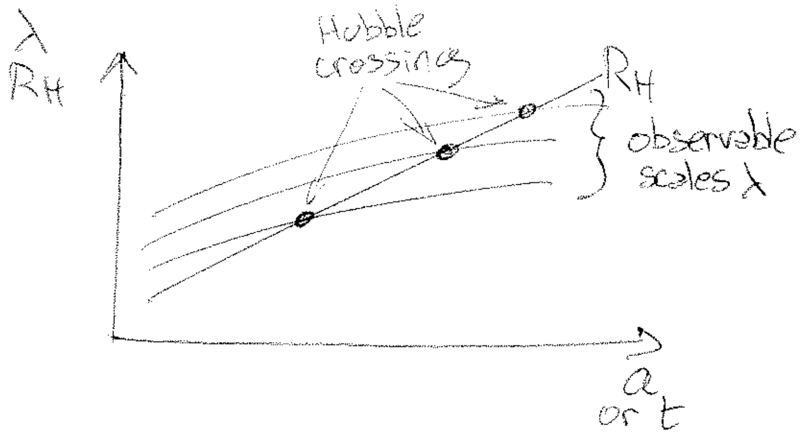
$$\lambda \simeq R_H \iff a\frac{2\pi}{k} \simeq \frac{1}{H} \iff k \simeq 2\pi a H$$

Usually one drops $2\pi$ factor and retain the definition: Hubble crossing $\iff k = aH$

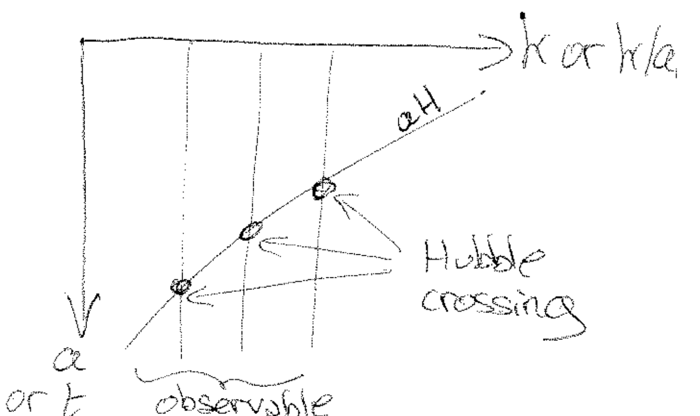
Note that $aH = \dot{a}$ decreases for decelerated expansion stages.

Diagrams of scales vs $R_H$ evolution:

physical distances:




inverse comoving distances (comoving wavenumbers):



→ remark 3: Theoretical predictions for spectrum of perturbations usually reported in following units: "$k$ in $h\,Mpc^{-1}$"

We understand from 1st remark that this in in fact $k/a_0$, but why put factor $h$ in definition of units?

⇝ this allows to incorporate trivial dependance on $H_0$ of the spectra. Indeed, evolution of each mode depends on comparision between $\lambda$ and only scale set by expansion: $R_H$. So the relevant ratio is

$$\frac{\lambda}{R_H} = \frac{a 2\pi}{k} H = 2\pi \frac{aH}{k} \sim \frac{aH}{k}$$

Today:

$$\frac{\lambda}{R_H} \sim \frac{a_0 H_0}{k} = \left(\frac{a_0 h}{k}\right)\left(\frac{H_0}{h}\right) \leftarrow \text{number } \left(= \frac{1}{3000} Mpc^{-1}\right)$$

So the trivial dependence of all spectra on $H_0$ can be absorbed by using units of $[h\,Mpc^{-1}]$ for $k/a_0$!