Chapter II Linearized gravity

References: $\begin{cases} \text{Bardeen 1980 PRD 22, 1882} \\ \text{Ma \& Bertschinger [astro-ph/9506072]} \end{cases}$

II.1. The gauge ambiguity

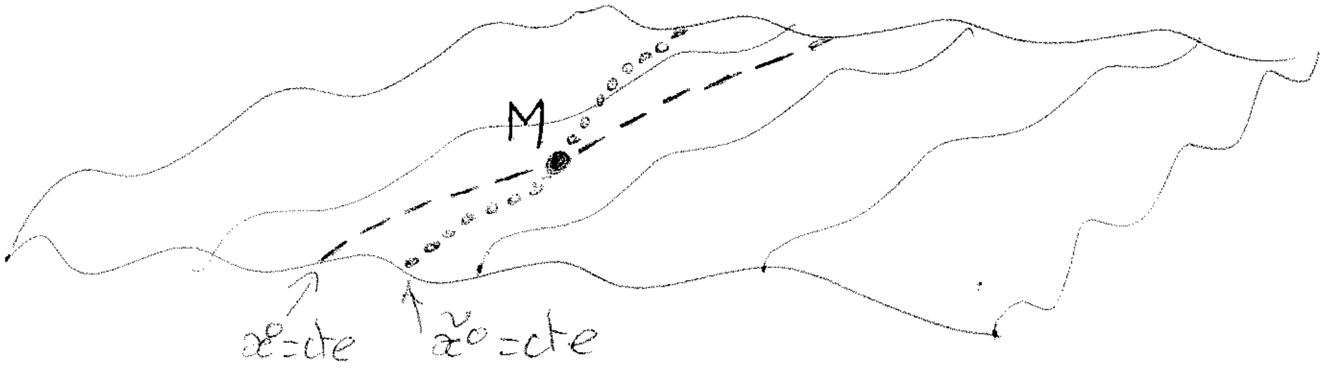
Decomposition in background + (small) perturbations

$$\begin{cases} g_{\mu\nu}(x^\alpha) = \bar{g}_{\mu\nu}(x^0) + \delta g_{\mu\nu}(x^\alpha) \\ T_{\mu\nu}(x^\alpha) = \bar{T}_{\mu\nu}(x^0) + \delta T_{\mu\nu}(x^\alpha) \end{cases}$$

physical quantities - Well-known effect of changing coordinates:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x)$$

spatial average over each time-slice. Since time-slicing is coordinate-dependent, this introduces some ambiguity in relation between $\bar{g}_{\mu\nu}$ and x^0, and in definition of $\delta g_{\mu\nu}$!



In system x, perturbation in M = quantity in M − {average along - - -}

" " $\tilde{x}$, " " M = quantity in M − {average along • • •}

→ a change of coordinates induces a different correspondance between perturbed and homogeneous quantities (but only if it changes the time-slicing).

→ some extreme change of time-slicing can lead to $\delta g_{\mu\nu} \gg \bar{g}_{\mu\nu}$ and $\delta T_{\mu\nu} \gg \bar{T}_{\mu\nu}$ (departure from linear perturbation theory): uninteresting. However, there exist an infinite number of "small" change of coordinates changing time-slicing and preserving smallness of perturbations: called "gauge transformation":
$$x^\mu \longmapsto \tilde{x}^\mu = x^\mu + \varepsilon^\mu$$

Simplistic example:

2D toy-universe described by $g_{\mu\nu}$ and ρ in coordinate system (t,x):

$$\begin{cases} ds^2 = dt^2 + 2\varepsilon \cos x \, dx\, dt + (\varepsilon^2 \cos^2 x - 2\varepsilon t \sin x - t^2) dx^2 \\ \rho = t + \varepsilon \sin x \qquad \text{(with } \varepsilon \text{ small)} \end{cases}$$

Seems to be a universe with FLRW background ($\bar{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -t^2 \end{pmatrix}$, $\bar{\rho} = t$) and small periodic perturbations.

We decide to change to coordinates (t', x'):

$$\begin{cases} t' = t + \varepsilon \sin x \\ x' = x \end{cases}$$

Then $dt' = dt + \varepsilon \cos x \, dx$, $dx' = dx$, so:

$$\leadsto ds^2 = (dt' - \varepsilon \cos x' \, dx')^2 + 2\varepsilon \cos x' dx' (dt' - \varepsilon \cos x' dx')$$
$$+ \left[\varepsilon^2 \cos^2 x' - \underbrace{(\varepsilon \sin x' + t)^2}_{t'^2}\right] dx'^2$$
$$= dt'^2 - t'^2 dx'^2$$

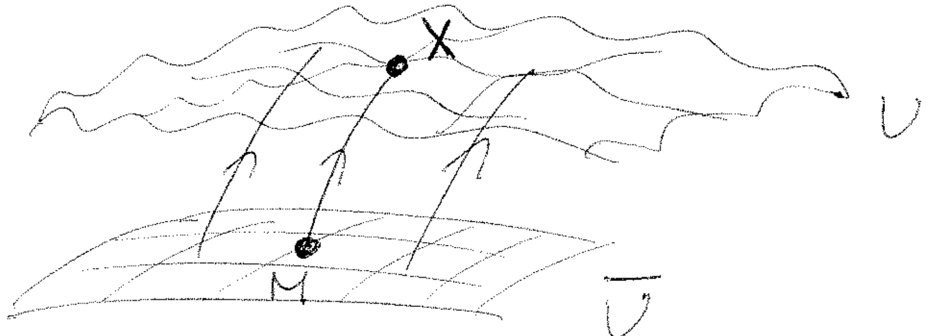
$\leadsto \rho = t'$

In new system, this universe looks like perfectly homogeneous FLRW! Perturbations "absorbed" in coordinate transformation.

In real 4D Universe: number of perturbed d.o.f. (degrees of freedom) are such that a gauge transf. can absorb some perturbations, but not all of them simultaneously.

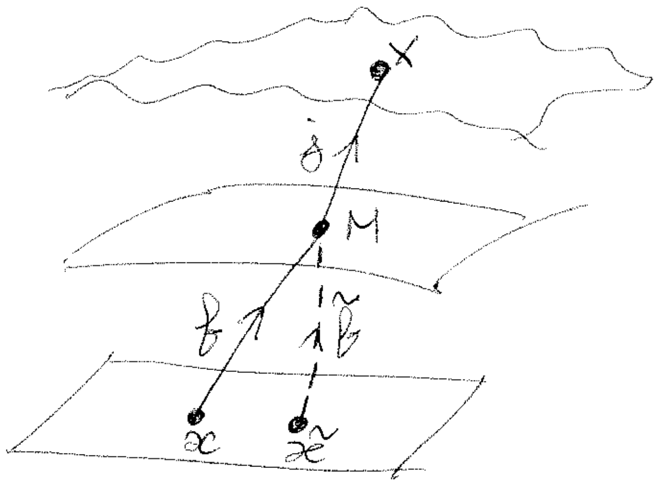
II.2) Mathematical definition of gauge transformation:

Let's call U the variety describing the real (perturbed) universe. Let's introduce a fictious homogeneous universe $\bar{U}$. A given gauge is a given mapping between U and $\bar{U}$.



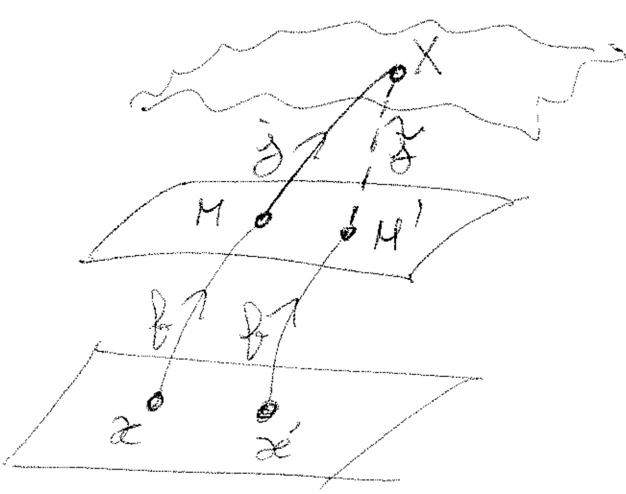
let us describe $\bar{U}$ with a system of coordinate (= mapping between $\mathbb{R}^4$ and $\bar{U}$).

⇝ a change of coordinates with fixed gauge is a change in mapping between $\mathbb{R}^4$ and U, but not U and $\bar{U}$:



Perturbation in x system = perturbation in the same point M in x' system (although coordinate of X and M changed)

⇝ a "pure" gauge transformation is the orthogonal transformation: change j while f is fixed:



After transformation, perturbation in X defined by comparing X with different point M' (although the mapping between $\mathbb{R}^4$ and $\bar{U}$ did not change).

Infinitesimal transformations of this second type are defined mathematically as "Lie derivation". The change from j to $\tilde{j}$ induces a vector field ε in $\mathbb{R}^4$: $\varepsilon^\mu = x'^\mu - x^\mu$ such that physical points associated previously to x^μ are now associated to x'^μ $\left(X = j \circ f(x^\mu) = j' \circ f(x'^\mu)\right)$.

2) <u>Gauge transformation / Lie derivative for Lorentz scalar</u>

Let $S(\vec{x})$ be a Lorentz scalar (application: in 1st order perturbation theory, the density $\rho(x)$ transforms like a Lorentz scalar, although formally $\rho(\vec{x}) = T^0_0(\vec{x})$)

Before gauge transformation, perturbation in point X given by: $\delta S(\vec{x}) = \underset{\substack{\uparrow \\ \text{value in } X}}{S(x^\mu)} - \underset{\substack{\uparrow \\ \text{value in } M}}{\bar{S}(x^\mu)}$

After gauge transformation, perturbation in same point X given by: $\delta S'(x'^\mu) = \underset{\substack{\uparrow \\ \text{value in } X}}{S'(x'^\mu)} - \underset{\substack{\uparrow \\ \text{value in } M'}}{\bar{S}'(x'^\mu)}$

* Since S is a Lorentz scalar, in a given physical point S is always the same number: $S(x^\mu) = S'(x'^\mu)$

* Since we did not change the mapping between $\mathbb{R}^4$ and $\bar{U}$, the functions S and $\bar{S}$ are identical

... so $\delta S'(x'^\mu) = S(x^\mu) - \bar{S}(x'^\mu)$

Now, we Taylor-expand at first order in $\varepsilon^\mu = x'^\mu - x^\mu$

$$\begin{aligned}\delta S'(x'^\mu) &= S(x'^\mu - \varepsilon^\mu) - \bar{S}(x'^\mu)\\ &= \underbrace{S(x'^\mu) - \bar{S}(x'^\mu)} - \varepsilon^\mu \partial_\mu S(x'^\mu)\\ &= \delta S(x'^\mu) - \varepsilon^\mu \partial_\mu S(x'^\mu)\end{aligned}$$

We infer the following equality (at the level of functions): $\delta S' = \delta S - \varepsilon^\mu \partial_\mu S$

Moreover: * ε^μ is 1st order in perturbations

* $\partial_0 S$ has a background component $\partial_0 \bar{S}$ and a 1st order component $(\partial_0 S - \partial_0 \bar{S})$

$i = 1,2,3$ = spatial indices * $\partial_i S$ is 1st order since $\partial_i \bar{S} = 0$ ($\bar{S}$ = homogeneous

... so $\varepsilon^i \partial_i S$ is 2nd order in perturbations.

At 1st order: $$\boxed{\delta S' = \delta S - \varepsilon^0 \partial_0 \bar{S}}$$ GAUGE TRANSFORMATION FOR LORENTZ SCALARS

b) Gauge transformation / Lie derivative for Lorentz vectors

... can be computed as an exercise

c) " " " " for Lorentz tensors

Similar in spirit to a) (see a) for more detailed explanations)

Before : $\delta T_{\mu\nu}(x^\mu) = T_{\mu\nu}(x^\mu) - \bar{T}_{\mu\nu}(x^\mu)$

After : $\delta T'_{\mu\nu}(x'^\mu) = T'_{\mu\nu}(x'^\mu) - \bar{T}'_{\mu\nu}(x'^\mu)$

$\rightarrow$ Lorentz tensor: $T'_{\mu\nu}(x'^\mu) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} T_{\alpha\beta}(x^\mu)$

$\rightarrow$ no coordinate transfo. on $\bar{U}$: $\bar{T} \equiv \bar{T}'$

$$\Rightarrow \quad \delta T'_{\mu\nu}(x'^\mu) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} T_{\mu\nu}(x^\mu) - \bar{T}_{\mu\nu}(x'^\mu)$$

Take $x^\mu = x'^\mu - \varepsilon^\mu$ and Taylor-expand in ε^μ:

$$\delta T'_{\mu\nu}(x'^\mu) = (\delta^\alpha_\mu - \partial_\mu \varepsilon^\alpha)(\delta^\beta_\nu - \partial_\nu \varepsilon^\beta)\left(T_{\mu\nu}(x'^\mu) - \varepsilon^\lambda \partial_\lambda T_{\alpha\beta}(x'^\mu)\right) - \bar{T}_{\mu\nu}(x'^\mu)$$

Keep only first order:

$$\begin{aligned}\delta T'_{\mu\nu}(x'^\mu) &= T_{\mu\nu}(x'^\mu) - \bar{T}_{\mu\nu}(x'^\mu) - (\partial_\mu \varepsilon^\alpha) T_{\alpha\nu}(x'^\mu) \\ &\quad - (\partial_\nu \varepsilon^\beta) T_{\mu\beta}(x'^\mu) - \varepsilon^\lambda (\partial_\lambda T_{\mu\nu})(x'^\mu) \\ &= \delta T_{\mu\nu}(x'^\mu) - (\partial_\mu \varepsilon^\alpha)\bar{T}_{\alpha\nu}(x'^\mu) - (\partial_\nu \varepsilon^\beta)\bar{T}_{\mu\beta}(x'^\mu) \\ &\quad - \varepsilon^0 (\partial_0 \bar{T}_{\mu\nu})(x'^\mu)\end{aligned}$$

Equality at the functional level:

At 1st order: $$\boxed{\delta T'_{\mu\nu} = \delta T_{\mu\nu} - (\partial_\mu \varepsilon^\alpha)\bar{T}_{\alpha\nu} - (\partial_\nu \varepsilon^\alpha)\bar{T}_{\mu\alpha} - \varepsilon^0(\partial_0 \bar{T}_{\mu\nu})}$$

{ GAUGE TRANSFORMATION FOR LORENTZ TENSORS exercise: the density is defined as $\rho = T^0_0$. Raise indices in previous result in order to show that at first order in perturbations, $\delta\rho$ transforms as if ρ was a scalar:

$$\delta\rho' = \delta\rho - \varepsilon^0 \partial_0 \bar{\rho} = \delta\rho - \varepsilon^0 \dot{\bar{\rho}}$$

II.3.) Classification of metric perturbations

$\delta g_{\mu\nu}$ is symmetric with 10 independent components.

Bardeen 1980 $\Rightarrow$ these 10 d.o.f. can be classified in scalars / vectors / tensors under 3D spatial rotations (not Lorentz transformations!)

$\Rightarrow$ these 3 sectors are NOT COUPLED at 1st order in perturbation theory (although Einstein equations are non-linear)

Most general metric perturbations read:

$$ds^2 = a^2(\tau)\left[(1+2\phi)d\tau^2 + B_i\, dx^i d\tau - \{(1-2\psi)\delta_{ij} + H_{ij}\} dx^i dx^j\right]$$

conformal time

sum over repeated indices

traceless: $H_i{}^i = 0$

where ϕ, B_i, ψ and H_{ij} depend on (τ, x^i)

B_i and H_{ij} can be further decomposed:

$$* \quad B_i = \underbrace{\partial_i b}_{\substack{\text{gradient} \\ \equiv \\ \text{longitudinal} \\ \text{part}}} + \underbrace{\varepsilon_{ijk}\, \partial_j b_k}_{\substack{\text{curl} \\ \equiv \\ \text{transverse} \\ \text{part}}}$$ with b_i divergenceless ($\partial_i b_i =$
(contains 2 d.o.f.)
degrees of freedom

$$* \quad H_{ij} = \underbrace{2\left(\partial_i \partial_j - \tfrac{1}{3}\delta_{ij}\Delta\right)\mu}_{\substack{\text{longitudinal} \\ \text{divergence} \\ \text{part}}} + \underbrace{\partial_i A_j + \partial_j A_i}_{\substack{\text{transverse} \\ \text{divergence} \\ \text{part}}} + \underbrace{H^T_{ij}}_{\substack{\text{transverse} \\ \text{part}}}$$ with $\forall j\ \partial_i H^T_{ij} = 0$

H_{ij} must be traceless:

- automatic for first part (this is the reason for the $\frac{1}{3}\delta_{ij}\Delta$)
- imposes $\partial_i A_i = 0$: A_i contains 2 d.o.f.
- imposes $H_i^{\ i} = 0$: H^T_{ij} contains $\underset{\substack{\text{symetric} \\ 3\times 3 \text{ tensor}}}{6} - \underset{\substack{\text{transverse} \\ \text{constraint}}}{3} - \underset{\substack{\text{traceless} \\ \text{constraint}}}{1} = 2$ d.o.f.

SCALAR SECTOR $\equiv \{\phi, \psi, b, \mu\} \rightarrow 1+1+1+1 = 4$ d.o.f.
VECTOR SECTOR $\equiv \{b_i, A_i\} \rightarrow 2+2 = 4$ d.o.f.
TENSOR SECTOR $\equiv H^T_{ij} \equiv$ gravitational waves $\rightarrow 2$ d.o.f.

The impact of gauge transformations can be computed from previous formula for tensors of Lorentz. E.g:

$$\delta g'_{00} = \delta g_{00} - 2\left(\partial_0 \varepsilon^\alpha\right) g_{\alpha 0} - \varepsilon^0 \partial_0 g_{00}$$

$$\Leftrightarrow \quad 2a^2\phi' = 2a^2\phi - 2\dot{\varepsilon}^0 a^2 - \varepsilon^0\, 2a\dot{a}$$ keeping only 1st order

In this way one can compute general transformation laws (induced by infinitesimal vector $\varepsilon^\mu(x^\nu)$):

$$\begin{cases} \phi' = \phi - \dot{\varepsilon}^0 - \frac{\dot{a}}{a}\varepsilon^0 \\ \psi' = \psi + \frac{1}{3}\nabla^2\beta + \frac{\dot{a}}{a}\varepsilon^0 \\ b' = b - \varepsilon^0 + \dot{\beta} \\ \mu' = \mu - \beta \end{cases} \quad \text{for scalars}$$

where we decomposed ε^i (spatial transformation) as:

$$\varepsilon^i \equiv \partial_i\beta + \varepsilon_{ijk}\partial_j\beta_k \quad \text{with } \partial_k\beta_k = 0$$

It is possible to derive similar laws for the vector components, and to show that the tensor component H_{ij}^T is gauge-invariant.

Gauge transformations have 4 d.o.f. (ε^μ). Two of them (ε^0 and β) can always cancel two scalar perturbations; the other two (β_k with $\partial_k\beta_k = 0$) can cancel two vector perturbations. Hence, the three sectors all contain two gauge-independent degrees of freedom.

For instance, (ϕ, ψ, b, μ) can be combined in two independent gauge-invariant quantities; a particular choice is the two "Bardeen potentials" Φ_A, Φ_H (see exercises).

In order to carry calculations, the simplest approach consists in fixing the gauge (i.e. use a prescription such that the time-slicing is fixed and two scalar metric perturbations vanish), studying/integrating the equations, and translate the final result in terms of gauge-invariant quantities. However, observable quantities are naturally gauge-invariant, because they refer to small distance scales where the time-slicing is not ambiguous. For instance, the total matter perturbation in two gauges ($\frac{\delta\rho_m^1}{\rho_m}$ and $\frac{\delta\rho_m^2}{\rho_m^2}$) are significantly different only on super-Hubble scales ($k < aH$), but we can observe them only for $k \gg aH$, so $\frac{\delta\rho_m}{\rho_m}$ can be computed in any gauge in order to derive the observable power spectrum $P(k) \equiv \langle |\frac{\delta\rho_m}{\rho_m}(k,\tau_0)|^2 \rangle$ measured for $k \gg a_0 H_0$ only.

In the rest of the course we will use the gauge-fixing prescription $\delta g_{0i} = \delta g_{i\neq j} = 0$ ($\Leftrightarrow \mu = B_i = 0$) which defines the "Newtonian" or "longitudinal" gauge. Then, scalar sector described by $ds^2 = a^2 [(1+2\Phi)d\tau^2 - (1-2\Psi)d\vec{x}^2]$

$\underset{\text{gravitational potential inside P.}}{|||}$

II.4. Linearized Einstein equations

*** scalar sector:** in Newtonian gauge, one obtains

(1) $\delta G^0_0 = 2a^{-2}\left\{-3\left(\frac{a'}{a}\right)^2\phi - 3\frac{a'}{a}\psi' + \Delta\psi\right\} = 8\pi G\,\delta T^0_0$

(2) $\delta G^0_i = 2a^{-2}\,\partial_i\left\{\frac{a'}{a}\phi + \psi'\right\} = 8\pi G\,\delta T^0_i$

(3) $\delta G^i_j = -2a^{-2}\Big\{\Big[\Big(2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\Big)\phi + \frac{a'}{a}(\phi' + 2\psi') + \psi'' + \frac{1}{2}\Delta(\phi - \psi)\Big]\delta^i_j$
$- \frac{1}{2}\partial^i\partial_j(\phi - \psi)\Big\} = 8\pi G\,\delta T^i_j$

with $' \equiv \frac{d}{d\tau}$ (conformal time).

From these equations one can read directly which components of δT^μ_ν are coupled with scalar perturbations

- δG^0_0 coupled with density perturbation $\delta\rho \equiv \delta T^0_0$
- δG^0_i " " component of the energy flux δT^0_i which is the gradient of some function v: $\delta T^0_i = \partial_i v$. Conventionally people use variable θ instead of v:

$$\theta \equiv \frac{\sum_i \partial_i\,\delta T^0_i}{(\bar\rho + \bar p)} = \frac{\Delta v}{\bar\rho + \bar p}$$ (θ called "velocity" by convention)

- δG^i_j couples with diagonal component of δT^i_j and with component such that $\delta T^i_j = \partial^i\partial_j s$. Alternatively this δT^i_j can be decomposed in pressure perturbation plus a traceless longitudinal divergence term Π^i_j:

$\delta T^i_j \equiv -\delta p\, \delta^i_j + \Pi^i_j$ ← traceless longitudinal divergence:

$$\Pi^i_j \equiv \underbrace{(\partial_i \partial_j - \tfrac{1}{3}\delta_{ij}\Delta)}_{\text{ensures that } \Pi_i{}^i = 0}\tilde{\sigma}$$

Conventionally people use the variable σ instead of $\tilde{\sigma}$:

$$(\bar{\rho}+\bar{p})\Delta\sigma \equiv -\sum_{i,j}(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\Delta)\Pi^i_j = -\tfrac{2}{3}\Delta(\Delta\tilde{\sigma})$$

σ is called the "shear" or "anisotropic stress".

Note that θ and σ are dimensionless. Physically, they can be seen as "potentials" giving raise to some energy flux (δT^i_0) and anisotropic pressure (Π^i_j).

Hence δT^μ_ν has 4 d.o.f. which can source scalar metric perturbations, through the 4 equations:

(4) $\quad -3\left(\frac{a'}{a}\right)^2\phi - 3\frac{a'}{a}\psi' + \Delta\psi = 4\pi G a^2 \delta\rho$

(5) $\quad \Delta\left(\frac{a'}{a}\phi + \psi'\right) = 4\pi G a^2(\bar{\rho}+\bar{p})\theta$

(6) $\quad \left(2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right)\phi + \frac{a'}{a}(\phi' + 2\psi') + \psi'' + \frac{1}{3}\Delta(\phi-\psi) = 4\pi G a^2 \delta p$

(7) $\quad -\Delta(\phi-\psi) = 12\pi G a^2(\bar{\rho}+\bar{p})\sigma$

(Note: (5) comes from $\partial_i(\delta G^0_i) = 8\pi G\, \partial_i(\delta T^0_i)$; (6) and (7) come from the decomposition of (3) into trace part and traceless longitudinal divergence part; for getting (7) one must apply the operator $\sum_{i,j}(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta)$ to this second part, and use the identity

$$\sum_{i,j}(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\Delta)(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\Delta)A = \tfrac{2}{3}\Delta(\Delta A);$$

this shows that the Laplacien of (7) is true; Hence (7) is true modulo a linear function of coordinates x^1, x^2, x^3 which must vanish, since it would diverge at infinity and violate the condition for small perturbations).

These equations can be used in order to prove that in absence of source ($\delta\rho = \theta = \delta p = \sigma = 0$), ϕ and ψ must either vanish, or be equal to a function of time which could be absorbed in a redefinition of the background. Hence the scalar metric perturbation do not propagate: they only follow scalar matter perturbations.

* **vector sector:** equations are easy to derive, but usually useless for standard cosmology (for "usual" source terms δT^{μ}_{ν}, the vector perturbations are driven to zero).

* **tensor sector:** the two degrees of freedom contained in H^T_{ij} (traceless transverse part of the metric) can be written as

$$H^T_{ij} = h_1 e^1_{ij} + h_2 e^2_{ij}$$

where h_1 and h_2 are functions of x^μ (the two degrees of polarisation of the graviton), and the two polarisation tensors $e^{1,2}_{ij}$ (independent of x^μ) are normalized in such way that

$$\begin{cases} \sum_{ij} e^1_{ij} e^1_{ij} = \sum_{ij} e^2_{ij} e^2_{ij} = \frac{1}{2} \quad \left(\text{so } \sum_{\lambda,i,j} e^\lambda_{ij} e^\lambda_{ij} = 1\right) \\ \sum_{ij} e^1_{ij} e^2_{ij} = 0 \text{ (orthogonal)} \end{cases}$$

Here we do not give the explicit expression of e^λ_{ij}. Starting from $\delta g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{a^2 H^T_{ij}} & \\ 0 & & & \end{pmatrix}$ one obtains the Einstein equations for tensor, which can be decomposed in two identical equations for h_1 and h_2:

$$h''_\lambda + 2\frac{a'}{a} h'_\lambda + \Delta h_\lambda = \text{source term}$$

(some components of δT^i_j)

Even when the source term vanishes, h_λ obey to a wave equation: tensor perturbations of the metric do propagate.