Chapter III Inflation

III.1) Motivations :

Historically, inflation introduced by Guth and by Starobinsky (independently in 1979) in order to solve several problems: flatness, monopoles, horizon.. Here we assume that these problems are known, but we review one of them in details:

- **Problem of causality:**

→ deceleraled expansion : take $a \propto t^n$, $n<1$, $\ddot{a}<0$

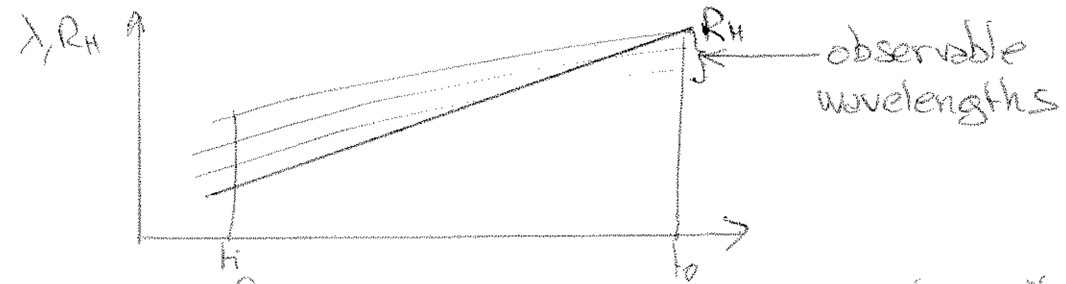
- wavelenghts $\lambda(t) = a(t)\frac{2\pi}{k}$ grow with $\ddot{\lambda}<0$
- causal horizon $d_H(t_1,t_2)$ grows linearily since:

$$d_H \equiv a(t_2)\int_{t_1}^{t_2}\frac{dt}{a(t)} = t_2^n\left[\frac{t^{1-n}}{1-n}\right]_{t_1}^{t_2} \xrightarrow[t_2 \gg t_1]{} \frac{t_2}{1-n} : \ddot{d}_H = 0$$

(remark: $R_H(t_2) = t_2/n$ so $d_H(t_2 \gg t_1) = \frac{n}{1-n}R_H(t_2)$)

We know that modes observable today are by construction such that $\lambda(t_0) \leq R_H(t_0)$ (see Chapter I.2.d))

So the picture is :



Inevitably, if we consider a small enough initial time t_i, all observable modes were acausal ($\lambda(t_i) \geq R_H(t_i)$).

Then there is no conceivable physical mechanism for generating these fluctuations.

This argument may sound a bit vague and loosy at this stage of the course. After understanding the physics and data on CMB fluctuations (Chapter V and beyond) it becomes obvious that in our Universe, observed fluctuations must have been initially in causal contact (we see non-trivial correlations in the large-wavelenght spectrum of CMB temperature and polarization fluctuations, showing that at the time of decoupling, modes with $\lambda(t_{dec}) \gg R_H(t_{dec})$ had been initially in causal contact).

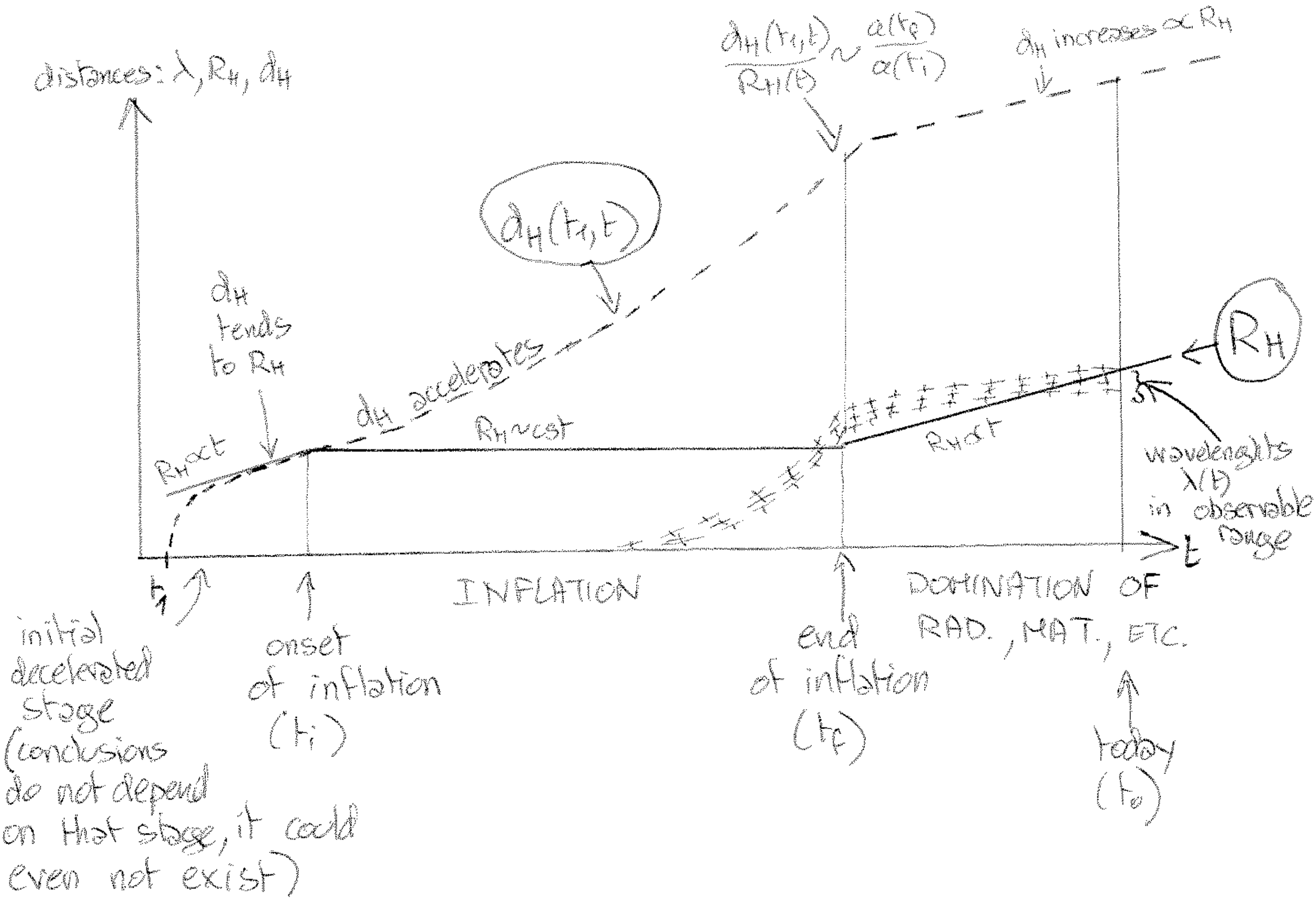
$\rightsquigarrow$ decelerated expansion: assume $\ddot{a} > 0$ (e.g. De Sitter)

- wavelenghts $\lambda(t) = a(t)\frac{2\pi}{k}$ grow with $\ddot{\lambda} > 0$
- causal horizon does not tend to R_H, but becomes arbitrarily larger if inflation is long enough.

  De Sitter example: $a \propto e^{Ht}$ $\quad H = \text{constant}$

  $$R_H = \frac{1}{H} \quad , \quad d_H = \frac{1}{H}\left(e^{H(t_2-t_1)} - 1\right) \longrightarrow \frac{1}{H} e^{H(t_2-t_1)} = R_H \frac{a(t_2)}{a(t_1)}$$

  So, for long enough inflation, we can always guarantee that all observable modes are in causal contact (see figure on next page).



The wavelenghts $\lambda(t)$ which we observe today on cosmic scale are in causal contact at all times during inflation and later. In addition, they satisfy $\lambda(t) \leq R_H(t)$ early enough during inflation. Each comoving scale $\frac{2\pi}{k}$:

* becomes super-Hubble during inflation when $\lambda \sim R_H$ ($\Leftrightarrow k \sim aH$)
* re-enters in Hubble radius later on
* is by construction such that $\lambda < R_H$ today (Chapter I)

$\rightsquigarrow$ definition of e-fold number: $N = \ln a$, $\Delta N = \ln \frac{a_2}{a_1}$

De Sitter $\rightarrow$ $N = Ht$

→ minimum duration of inflation for solving the problem of causality:

We assume that inflation = exact De Sitter stage (other wise we will find a stronger condition. We want the minimal duration, ie weakest condition).

Then, the problem is solved if at the beginning of inflation,

$$\lambda_{\substack{\text{observable}\\ \text{Universe}}}^{(t_i)} \leq R_H(t_i) \sim d_H(t_* \ll t_i, t_i)$$

To get minimal duration we saturate this bound:

$$\lambda_{\substack{\text{obs.}\\ \text{universe}}}(t_i) = R_H(t_i)$$

We use the fact that today:

$$\lambda_{\substack{\text{obs}\\ \text{universe}}}(t_0) = R_H(t_0) \qquad \text{with } \lambda_{\substack{\text{obs}\\ \text{univ}}}(t_0) = \frac{a(t_0)}{a(t_i)} \lambda_{\substack{\text{obs}\\ \text{univ}}}(t_i)$$

So $\displaystyle \frac{a(t_0)}{a(t_i)} = \frac{R_H(t_0)}{R_H(t_i)}$

But $R_H(t_i) = R_H(t_f)$ (exact De Sitter)

and $\displaystyle \frac{R_H(t_0)}{R_H(t_f)} = \left(\frac{a(t_0)}{a(t_f)}\right)^2$ assuming for <u>simplicity</u> radiation domination between t_f and t_0 (otherwise: small correction factor). Then,

$H^2 \sim \rho_{rad} \sim a^{-4} \Rightarrow H \sim a^{-2} \Rightarrow R_H \sim a^2$

So $\displaystyle \frac{a(t_0)}{a(t_i)} = \left(\frac{a(t_0)}{a(t_f)}\right)^2 \Longleftrightarrow \frac{a(t_f)}{a(t_i)} = \frac{a(t_0)}{a(t_f)}$

In terms of e-folds, this becomes:

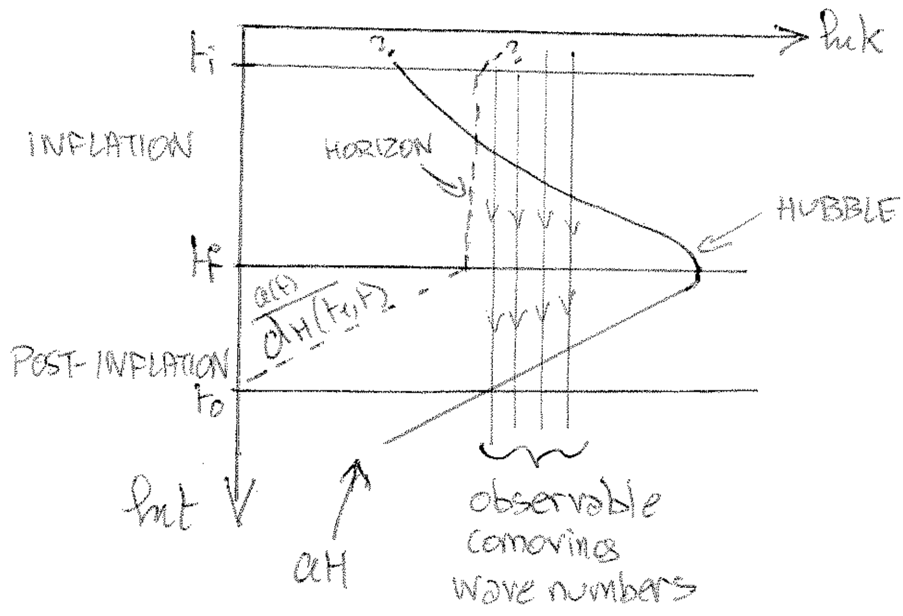
$$\underbrace{N_f - N_i}_{\Delta N_{\text{inflation}}} = \underbrace{N_0 - N_f}_{\Delta N_{\text{post-inflation}}}$$

In conclusion, the minimal duration of inflation is given by $\boxed{\Delta N_{\text{inflation}} \geq \Delta N_{\text{post-infation}}}$

number depending
on $\ln(a_0/a_f)$,
ie on $\ln(\rho_0/\rho_f)$,
ie on energy scale of inflation
(GUT scale $\to \Delta N_{\text{post-inflation}} \sim 60$)

- other problems (flatness, monopoles, homogeneity of CMB) are solved with roughly the same condition.

- summary in (k, t) space: (k: comoving wavenumber)



Hubble crossing:

$\lambda \sim R_H$

$\Leftrightarrow k \sim aH$

Horizon crossing:

$\lambda(t) \sim d_H(t_1, t)$

$\Rightarrow k \sim a(t)/d_H(t_1,t)$