III.2. Slow-roll conditions

In general, for background fluid/species with density ρ and pressure p, acceleration given by:

$$\left\{\begin{array}{l} \text{Friedmann: } (\dot{a}/a)^2 = \frac{8\pi G}{3}\rho \\ \text{Conservation: } \dot{\rho} = -3(\dot{a}/a)(\rho+p) \end{array}\right\} \Rightarrow \ddot{a} = -\sqrt{\frac{2\pi G}{3}}\frac{\dot{a}}{\sqrt{\rho}}(\rho+3p)$$

Expanding universe with $\ddot{a}>0$ requires $\underline{\rho+3p<0}$

Impossible with radiation, matter.

Possible with Λ but inflation would never end...

We try with a scalar field:

$$\mathcal{L}_\varphi = \frac{1}{2} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)$$

$$\Rightarrow T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \mathcal{L}_\varphi g_{\mu\nu}$$

Assume nearly homogeneous field $\varphi(\vec{x},t) = \bar{\varphi}(t) + \delta\varphi$

For the background part:

$$\rho = T^0_0 = \frac{1}{2}\dot{\bar{\varphi}}^2 + V(\bar{\varphi})$$

$$p = -T^i_i = \frac{1}{2}\dot{\bar{\varphi}}^2 - V(\bar{\varphi})$$

$$(\rho+3p)<0 \Leftrightarrow \dot{\bar{\varphi}}^2 < V(\bar{\varphi})$$

In order to ensure a long stage of inflation, we must impose:

$$\boxed{\dot{\bar{\varphi}}^2 \ll V(\bar{\varphi}) \quad 1^{st} \text{ Slow-roll condition}}$$

This should hold for extended period of time, so

$$|(\dot{\bar{\varphi}}^2)^\circ| \ll |(V)^\circ| \Leftrightarrow |\dot{\bar{\varphi}}\ddot{\bar{\varphi}}| \ll |\dot{\bar{\varphi}}\frac{\partial V}{\partial\varphi}|, \text{ so:}$$

$$\boxed{|\ddot{\bar{\varphi}}| \ll |\frac{\partial V}{\partial\varphi}(\bar{\varphi})| \quad 2^{nd} \text{ Slow-roll condition}}$$

During inflation, we can use simplified equations of motion:

① Friedmann: $G^0_0 = 8\pi G\, T^0_0 \Rightarrow 3H^2 = 8\pi G \rho = 8\pi G \left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)$

$$\text{Slow-Roll (SR)} \longrightarrow \boxed{3H^2 \simeq 8\pi G\, V(\varphi)}$$

② Klein-Gordon: $\dot{G}^0_0 + 3H(G^0_0 - G^i_i) = \dot{T}^0_0 + 3H(T^0_0 - T^i_i) = 0$ (Bianchi Identity)

$$\Leftrightarrow \ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V}{\partial \varphi}(\varphi) = 0$$

$$\text{Slow-Roll (SR)} \longrightarrow 3H\dot{\varphi} + \frac{\partial V}{\partial \varphi}(\varphi) \simeq 0$$

$$\Rightarrow \boxed{\dot{\varphi} \simeq -\frac{\partial V/\partial \varphi}{3H}}$$

Remarks:

* different authors use different definitions of SR conditions: $\dot{\varphi} \ll \# V$ and $|\ddot{\varphi}| \ll \# \left|\frac{\partial V}{\partial \varphi}\right|$

number of $\mathcal{O}(1)$ depending on authors

* SR conditions can be written differently, making use of Friedmann and Klein-Gordon (KG):

→ condition on $H(t)$: $\boxed{-\dot{H} \ll \# H^2 \text{ and } |\ddot{H}| \ll \# H^3}$

→ condition on $V(\varphi)$: $\boxed{M_P^2 \left(\frac{V'}{V}\right)^2 \ll \# \text{ and } M_P^2 \left|\frac{V''}{V}\right| \ll \#}$

with $' \equiv \frac{\partial}{\partial \varphi}$

→ Liddle & Lyth: $\underline{\varepsilon \equiv \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \quad \eta = \frac{1}{8\pi G}\frac{V''}{V}}$

SR: $\varepsilon \ll 1, \quad |\eta| \ll 1$

* Friedman + KG ⇒ EXACT RELATION $\boxed{\dot{H} = -4\pi G \dot{\varphi}^2 \leq 0}$