III.3. Quantization and semi-classical transition

In III.5, we will quantize the field and scalar metric perturbations.

In III.4., we will quantize the tensor metric perturbations.

Here we want to summarize the general principle of the calculation, and explain why at the end of inflation, we can treat the quantized perturbation $\delta\chi$ like <u>gaussian stochastic numbers</u> entirely described by the <u>Fourier spectrum</u> $\langle|\delta\chi_k|^2\rangle$, which can be inferred from <u>commutation relations</u> of quantum mechanics.

a) quantization of harmonic oscillator in flat space-time:

Classical equation of motion : $\ddot{x} + \omega^2 x = 0$

Fundamental state: gaussian wave function $\psi_0 = c^{st} e^{-\frac{\omega x^2}{2}}$

Probability of x in fundamental state:

$$P(x) = |\psi_0(x)|^2 = |c^{st}|^2 e^{-\omega x^2}$$

$$= \text{gaussian of variance } \langle x^2\rangle^{1/2} = \sigma = \sqrt{\frac{1}{2\omega}}$$

b) free massless scalar field in flat space-time:

Real space: $\mathcal{L} = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi \Rightarrow \Box\chi = 0 \quad (\chi\in\mathbb{R})$

Fourier space: $\begin{cases} \chi_k = \chi^*_{-k} \text{ since } \chi\in\mathbb{R} \\ \forall k,\ \ddot{\chi}_k + k^2\chi_k = 0 \end{cases}$

So, each mode k = independent harmonic oscillator

with:
- wave functional $\Psi_0(\chi_k) = \mathcal{N} e^{-\frac{1}{2}k|\chi_k|^2}$
- probability $P(\chi_k) = |\mathcal{N}|^2 e^{-k|\chi_k|^2}$
- variance $\langle|\chi_k|^2\rangle^{1/2} = \sigma = \sqrt{\frac{1}{2k}}$

Definition of the "mode function":

Equation $\ddot{\chi}_k + k^2\chi_k = 0 \Rightarrow$ ensemble of classical solutions

$$\chi_{\vec{k}} = A_{\vec{k}}\, e^{-ikt} + B_{\vec{k}}\, e^{ikt}$$

So each pair $(A_{\vec{k}}, B_{\vec{k}})$ defines a possible classical trajectory. One of them plays a special role from the point of view of the quantum system and is called the mode function.

Mode function = classical trajectory accounting for "typical evolution" of quantum system = evolution of $\chi_{\vec{k}}(t)$ corresponding to one standard deviation in the fundamental state: $\langle 0|\hat{\chi}_k \hat{\chi}_k^\dagger|0\rangle^{1/2}$

The mode function is found by imposing two conditions on the general classical solutions:

(1) Positive frequency solution:

Flat space-time $\Rightarrow \frac{\partial}{\partial t}$ = Killing vector (symmetry of spacetime)

$\Rightarrow$ some solutions are eigenfunctions of $\frac{\partial}{\partial t}$ operator:

$$\frac{\partial}{\partial t}\cancel{\chi_k} = -ik\chi_k$$

If $k>0$: positive frequency solution (physical solution)

② Solution normalized to commutation relation

$$[\hat{x}_k, \hat{p}_{x_k}] = i\hbar$$ later we take $\hbar = c = 1$

$\hat{x}_k$: "position"; $\hat{p}_{x_k}$: conjugate momentum

can be translated as a condition on the Wronskian:

$$\chi_k \dot{\chi}_k^* - \dot{\chi}_k \chi_k^* = i$$

Mode function in the problem at hand

$$\chi_{\vec{k}} = A_{\vec{k}} e^{ikt} + B_{\vec{k}} e^{-ikt}$$

$$\dot{\chi}_{\vec{k}} = ik A_{\vec{k}} e^{ikt} - ik B_{\vec{k}} e^{-ikt}$$

$$\begin{cases} \text{positive frequency} \Rightarrow B_{\vec{k}} = 0 \\ \text{Wronskian condition} \Rightarrow |A_{\vec{k}}| = \sqrt{\frac{1}{2k}} \end{cases}$$

So the mode function is defined (up to an arbitrary phase) as $\chi_k = \sqrt{\frac{1}{2k}}\, e^{-ikt}$

As expected, $|\chi_k| = \frac{1}{\sqrt{2k}}$ is equal to the variance $\sigma = \sqrt{\frac{1}{2k}}$ of the probability $P(\chi_k)$ for the fundamental state.

c) free massless field in curved space-time:

In general, curvature makes problem complicated or even ill-defined. $\frac{\partial}{\partial t}$ is not a Killing vector, so no positive frequency solution, no definition of fundamental state.

At time t_1 we can formally define annihilation/creation operators and Fock space. But Bogolioubov transformation transforms $\hat{a}(t_1)$ into a mixture of $\hat{a}(t_2)$ and $\hat{a}^\dagger(t_2)$. So fundamental state at time $t_1 \longrightarrow$ excited state at t_2. So we cannot previledge a particular definition of the fundamental state.... (see e.g. book by Birrell & Davies)

But this problems are evaded in the context of inflation!

Definition of fundamental state ("in" vacuum)

For quantizing a given mode k, we can start when $\lambda \ll R_H$, ie $k \gg aH$. Then, curvature is negligible, mode sees Minkowski space. Initial Fock space and vacuum state defined in usual way.

Remarks: * later, when $\lambda \gg R_H$, there will be "particle creation from the vacuum", although the evolution is unitary. Well-know effect of curved space-times!

* the problem is that initially, we can have $\frac{k}{aH} \ll 1$ but not $\frac{k}{aH} = 0$, so the field does see a tiny curvature, introducing a residual ambiguity in the definition of the vacuum. This lead to the speculation that there could be "transplanckian effects" altering slightly the standard computation presented thereafter.

Squeezing of the quantum state:

Initially, the mode sees Minkowski and has the wave functional discussed before (in the vacuum state): $\Psi_0(\chi_k) = \mathcal{N} e^{-\frac{1}{4}\left(\frac{|\chi_k|^2}{\sigma_k^2}\right)}$ with $\sigma_k = \frac{1}{\sqrt{2k}}$

After horizon crossing, it is possible to show that $\Psi_0(\chi_k) = \mathcal{N} \exp\left\{-\frac{1}{4} \frac{|\chi_k|^2}{\sigma_k(t)^2} \left[1 + i F_k(t)\right]\right\}$

with: $\begin{cases} \sigma_k(t) = \text{mode function (starts from } \sqrt{\frac{1}{2k}}\text{, then evolves according to equation of motion)} \\ |F_k(t)| \text{ goes from } 0 \text{ (limit } \lambda \ll R_H) \text{ to } \infty \; (\lambda \gg R_H) \end{cases}$

This evolution is well-known from experts in quantum mechanics: when $|F_k| \to \infty$ the state is called a "squeezed state" and is known to be indistinguishable from a classical stochastic system with a classical distribution of probability

$$P(x_k) = |\psi_0(x_k)|^2 = |c|^2 \exp\left\{-\frac{1}{2}\frac{|x_k|^2}{\sigma_k(t)^2}\right\}$$

There are various interpretations of this "quantum to semi-classical transition":

semi means here stochastic...

(1) In QFT, $\hat{x}_k$ and $\hat{p}_{x_k}$ do not commute. We could neglect this non-commutation provided that it changes expectation values by a tiny amount. In particular, if:

$$\langle 0|\{\hat{x}_k, \hat{p}_{x_k}\}|0\rangle \gg \langle 0|[\hat{x}_k, \hat{p}_{x_k}]|0\rangle \quad (a)$$

(<anti-commutator> >> <commutator>)

then it is clear that all expectation values can be computed as if $[\hat{x}_k, \hat{p}_{x_k}] \simeq 0$, ie in a classical way. So (a) is a condition for semi-classical transition.

Precisely, for a squeezed state, (a) is true whenever $|F_k(t)| \to \infty$!!

(2) Remember that $\hat{N}_{\vec{k}} = \int d^3k\, \hat{a}_{\vec{k}}^{\dagger} \hat{a}_{\vec{k}}$ is the "occupation number operator" for each mode. But $\hat{a}_k$ can be written in terms of $\hat{x}_k$ and $\hat{p}_k$. Then, one can show that (a) is equivalent to $\langle 0 | \hat{N} | 0 \rangle \gg 1$, ie to an occupation number $\gg 1$. Again, $\langle 0 | \hat{N} | 0 \rangle \gg 1$ is true in the limit $|F_k(t)| \longrightarrow \infty$.

Many people use the fact that "when the occupation number is very large, the system is <u>effectively</u> classical".

What should one rember from section III.3 ?

<u>FIRST</u> Quantification of a field in inflationnary background is POSSIBLE because each mode is initially sub-Hubble. However, horizon crossing ($k < aH$) will introduce at some point some non-trivial effects, namely "particle creation from the vacuum" ($\langle 0 | \hat{N} | 0 \rangle \longrightarrow \infty$) and a "semi-classical transition" (= system equivalent to classical stochastic system).

$\boxed{\begin{array}{l}\underline{\text{SECOND}} \quad \text{The semi-classical system is described by a gaussian distribution} \\ \text{of probability with a variance given by the mode function.}\end{array}}$

Consequence: at the end of inflation, we can forget everything about quantization. We can describe everything with the Fourier power spectrum

$\langle |\chi_k|^2 \rangle = \text{squared variance} = \sigma_k^2(t)$

However, the quantum origin of the fluctuations is manifest:

* in the fact that the classical probability is a gaussian: comes from gaussian wave function of vacuum state of quantum field modes.
* in the fact that $\sigma_k(t)$ is found by computing the <u>mode function</u>, which is normalized initially on the basis of quantum mechanics (positive frequency, Wronskian $= i\,(\hbar)$)