III.4. Tensor perturbations:

We remember from Chapter II that $\delta g_{\mu\nu}$ contains:

$$\delta g_{ij} = -a^2 \underbrace{H^T_{ij}}_{\text{traceless transverse}} + \text{other terms}$$

with $H^T_{ij} = h_1 e^1_{ij} + h_2 e^2_{ij}$ (2 degrees of polarization of the graviton)

$h_{1,2}(x^\mu)$ = two fields describing gravitational waves

$e^{1,2}_{ij}$ = two fixed tensors (traceless, transverse) normalized to $\sum_{i,j} e^\lambda_{ij} e^{\lambda'}_{ij} = \frac{\delta_{\lambda\lambda'}}{2}$

(So $\sum_{\lambda,\lambda',i,j} e^\lambda_{ij} e^{\lambda'}_{ij} = \frac{1}{2} + \frac{1}{2} = 1$)

Equation of motion $\quad h_\lambda'' + 2\frac{a'}{a} h_\lambda' + k^2 h_\lambda = 0$

($' = \frac{\partial}{\partial \tau}$ = derivative w.r.t. conformal time τ).

By going back to the action $S = \int d^4x \sqrt{|g|}\, \frac{R}{16\pi G}$ and expanding R in terms of h_1, h_2, it is possible to prove that the conjugate momentum of h_λ is $\mathbb{P}_{h_\lambda} = \frac{a^2}{64\pi G} h_\lambda'$ (ie $S = \int d\tau d^3x^i \, \frac{1}{2}\left[\frac{a^2}{64\pi G} {h'_\lambda}^2 + \dots\right]$)

■ mode function

We have all the elements in our hand for computing the mode function. (For simplicity, we will do the exact computation in the De Sitter limit.)

First, we change variable from h_λ to g, in order to eliminate the friction term in the equation of motion, and for simplicity to have a canonical kinetic term for g: with $g = \frac{a}{\sqrt{64\pi G}} h_\lambda$, we have for each Fourier mode $\vec{k}$:

$$\begin{cases} h_\lambda'' + 2\frac{a'}{a} h_\lambda' + k^2 h_\lambda = 0 \implies g'' + \left(k^2 - \frac{a''}{a}\right) g = 0 \\ [h_\lambda, p_\lambda] = \frac{a^2}{64\pi G}\left(h_\lambda h_\lambda^{*\prime} - h_\lambda^* h_\lambda'\right) = i \implies g g'^* - g^* g' = i \end{cases}$$

We note that in a De-Sitter or quasi-De-Sitter background, $\frac{a''}{a} \sim a^2 H^2$ (see proof below for De Sitter). So, the sub-Hubble limit $k \gg aH$ corresponds to $k^2 \gg \frac{a''}{a}$. In this limit, $g'' + k^2 g = 0$: the field g behaves like a canonically normalized scalar field in flat space-time, with the usual mode function $g = \frac{1}{\sqrt{2k}} e^{-ik\tau}$

- exact De Sitter limit

To know the solution at each stage we go to the limit of exact De Sitter: $a = \alpha e^{H_i t}$

arbitrary normalization factor

constant Hubble parameter during inflation

$d\tau \equiv \frac{dt}{a} = \alpha^{-1} e^{-H_i t} dt$, so up to a constant, we have $\tau = \alpha^{-1} \frac{e^{-H_i t}}{-H_i} = -\frac{1}{a H_i} \iff a = -\frac{1}{H_i \tau}$

So, during exact De Sitter, $a = -\frac{1}{H_i \tau}$ goes from 0 to ∞ when τ goes from $-\infty$ to 0...
(For a finite amount of inflation, one can choose the constant of integration so that $a = -\frac{1}{H_i \tau - \frac{1}{a_f}}$ but for simplicity we neglect this complication and adopt $a = -\frac{1}{H_i \tau}$).

Then $\frac{a'}{a} = -\frac{1}{\tau}$ and $\frac{a''}{a} = \frac{2}{\tau^2}$.
Note that $aH = -\frac{1}{H_i \tau} H_i = -\frac{1}{\tau}$, so $\frac{a''}{a} \sim a^2 H^2$ as expected.
The equation for φ now reduces to:

$$\varphi'' + \left(k^2 - \frac{2}{\tau^2}\right)\varphi = 0 \quad \text{(for each mode } \vec{k}\text{)}.$$

Analytic solution given by Hankel functions (comb. of Bessel functions) of order $\frac{3}{2}$:

$$\varphi(\vec{k}) = A_{\vec{k}} \left(1 - \frac{i}{k\tau}\right) e^{-ik\tau} + B_{\vec{k}} \left(1 + \frac{i}{k\tau}\right) e^{ik\tau}$$

In order to obtain the correctly normalized mode function in the limit $k \gg aH \Leftrightarrow k\tau \ll -1$, we should take $B_{\vec{k}} = 0$ and $A_{\vec{k}} = \sqrt{\frac{1}{2k}}$. Then,

$$\varphi = \sqrt{\frac{1}{2k}} \left(1 - \frac{i}{k\tau}\right) e^{-ik\tau} \xrightarrow[(k\tau \ll -1)]{k \gg aH} \sqrt{\frac{1}{2k}}\, e^{-ik\tau}$$

This is the mode function for φ. For the field

$h_\lambda = \frac{\sqrt{16\pi G}}{a} g$, the mode function is then:

$$h_\lambda = \frac{\sqrt{16\pi G}}{a}\sqrt{\frac{1}{2k}}\left(1-\frac{i}{k\tau}\right)e^{-ik\tau} = \sqrt{\frac{32\pi G}{k^3}}\, H_i(i-k\tau)e^{-ik\tau}$$

For outside the Hubble radius ($k \ll aH \Leftrightarrow -1 \ll k\tau < 0$) we have:

$$h_\lambda \xrightarrow[(k \ll aH)]{k\tau \to 0} \sqrt{\frac{32\pi G}{k^3}}\, i H_i$$

We did all this work because we know that outside the Hubble radius, the squared modulus of the mode function corresponds to the squared variance of h_λ, seen now as a classical stochastic number:

$$\langle |h_\lambda(\vec{k})|^2 \rangle = \frac{32\pi G}{k^3} H_i^2$$

For each mode $\vec{k}$, the variance of gravitationnal waves (including all degrees of polarization) is then:

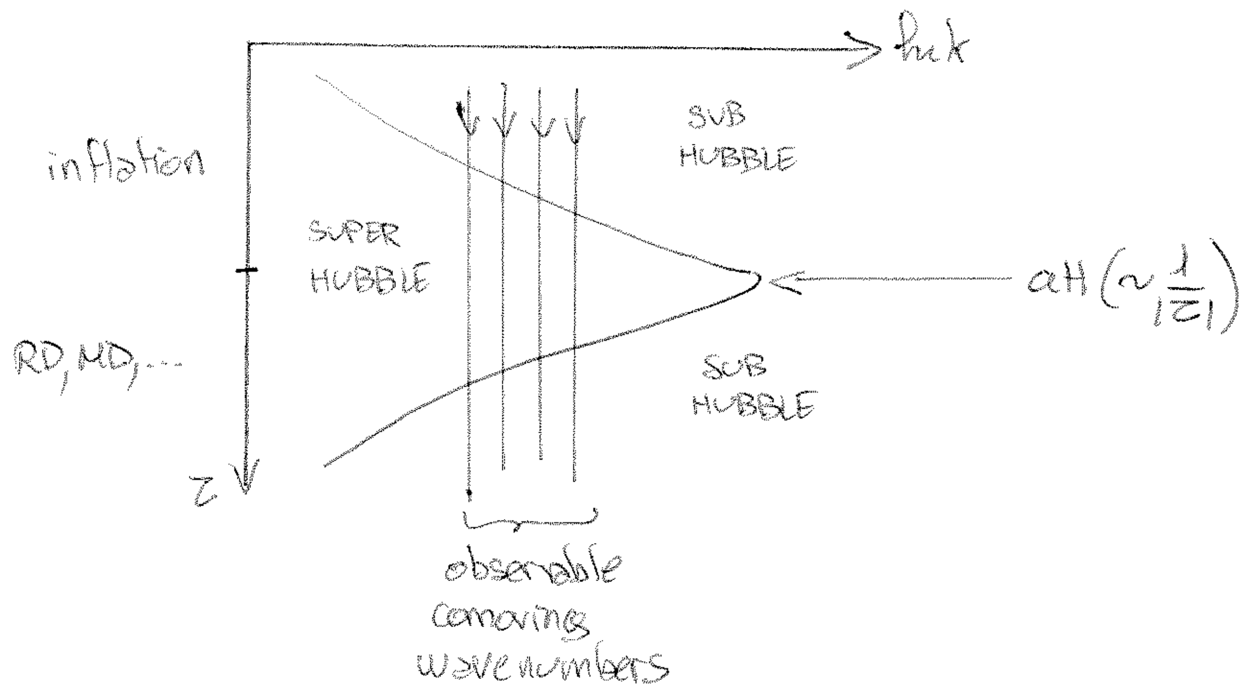
$$\sum_{i,j} \langle |H^T_{ij}|^2 \rangle = \langle |h_1|^2 \rangle \sum_{i,j} e^1_{ij} e^1_{ij} + \langle |h_2|^2 \rangle \sum_{i,j} e^2_{ij} e^2_{ij}$$

$$= \frac{1}{2}\langle |h_1|^2 \rangle + \frac{1}{2}\langle |h_2|^2 \rangle$$

$$= \frac{32\pi G}{k^3} H_i^2$$

Evolution des modes après l'inflation

Les equations du mouvement donnent les solutions (classiques) pour l'Evolution des modes de Fourier à tout moment:



In the super-Hubble region, the equation reduces to (both during inflation and after inflation):

$$y - \frac{a''}{a} y = 0 \Rightarrow y \propto a \text{ or } y \propto a \int \frac{d\tau}{a^2}$$

$$\Rightarrow h = \text{constant} \text{ or } h \propto \int \frac{d\tau}{a^2}$$

So, there is one constant and one decaying solution, which becomes quickly negligible. Hence, h is constant between the end of inflation, and the time at which the mode is about to re-enter the Hubble radius during RD/MD (After horizon crossing, the equation of motion tends to $h_\lambda'' + k^2 h_\lambda = 0$, and the solution is oscillatory, as expected for gravitationnal waves).

So, the quantity $\sum_{ij} \langle |H_{ij}^T|^2 \rangle$ computed at the end of inflation is very useful: it gives the correct initial conditions for the evolution of gravitationnal waves (GWs) during RD, MD, ...

notion of power spectrum

In this section we go back to general definitions. Our convention for 3D Fourier expansion is:

$$\begin{cases} \delta(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} \, \delta(\vec{k}) \, e^{i\vec{k}\vec{x}} \\ \delta(\vec{k}) = \int d^3\vec{x} \; \delta(\vec{x}) \, e^{-i\vec{k}\vec{x}} \end{cases}$$

perturbation $\delta(\vec{x}), \delta(\vec{k})$ = stochastic number; $\delta(\vec{x}) \in \mathbb{R} \Rightarrow \delta(\vec{k}) = \delta(-\vec{k})^*$

In real space, the two-point correlation function $\xi = \langle \delta(\vec{x}) \delta(\vec{x}+\vec{r}) \rangle$ does not depend on $\vec{x}$ (if the universe is homogeneous) nor on the direction $\hat{r} = \vec{r}/r$ (if the universe is isotropic):

$$\langle \delta(\vec{x}) \delta(\vec{x}+\vec{r}) \rangle = \xi(r)$$

If $\delta(\vec{x})$ is gaussian, $\xi(r)$ contains all the information about a given cosmological model (at the level of linear perturbations).

The power spectrum is the equivalent of $\xi(r)$ in Fourier space. Let us compute:

$$\langle \delta(\vec{k})\,\delta^*(\vec{k}')\rangle = \int d^3\vec{x}\, d^3\vec{x}'\ \langle \delta(\vec{x})\,\delta(\vec{x}')\rangle\, e^{-i(\vec{k}\vec{x}-\vec{k}'\vec{x}')}$$

$$= \int d^3\vec{x}\, d^3\vec{r}\, \langle \delta(\vec{x})\,\delta(\vec{x}+\vec{r})\rangle\, e^{-i(\vec{k}-\vec{k}')\vec{x}}\, e^{i\vec{k}'\vec{r}} \quad (\vec{x}'=\vec{x}+\vec{r})$$

$$= \int d^3\vec{r}\, \xi(r)\, e^{i\vec{k}'\vec{r}} \underbrace{\int d^3\vec{x}\, e^{-i(\vec{k}-\vec{k}')\vec{x}}}_{(2\pi)^3\,\delta(\vec{k}-\vec{k}')}$$

$$= (2\pi)^3 \underbrace{\left[\int d^3\vec{r}\, \xi(r)\, e^{i\vec{k}\vec{r}}\right]}\, \delta^{(3)}(\vec{k}-\vec{k}')$$

3D Fourier transform of $\xi(r)$, equal to

$$\int r^2 dr\, \sin\theta\, d\theta\, d\varphi\ \xi(r)\, e^{ikr\cos\theta} = 4\pi \int r^2 dr\ \xi(r)\, \frac{\sin kr}{kr}$$

↳ depends on k, not on $\vec{k}/k$ due to isotropy

Conversely, the real space correlation function is given in terms of Fourier perturbations by:

$$\langle \delta(\vec{x})\,\delta(\vec{x}+\vec{r})\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3}\, \langle \delta(\vec{k})\,\delta^*(\vec{k}')\rangle\, e^{i(\vec{k}-\vec{k}')\vec{x}}\ e^{-i\vec{k}'\vec{r}}$$

But $\langle \delta(\vec{k})\,\delta(\vec{k}')^*\rangle$ vanishes for $\vec{k}\neq\vec{k}'$ (as found in the previous calculation) and depends on k, not $\vec{k}/k$ (consequence of isotropy): so $\langle \delta(\vec{k})\,\delta^*(\vec{k}')\rangle = \underbrace{\langle|\delta(\vec{k})|^2\rangle}_{\text{function of } k \text{ only}}\ \delta^{(3)}(\vec{k}-\vec{k}')$

Then $$\langle \delta(\vec{x})\,\delta(\vec{x}+\vec{r})\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\, \langle|\delta(\vec{k})|^2\rangle\, e^{-i\vec{k}\vec{r}}$$

$$= \int \frac{k^2 dk\, \sin\theta\, d\theta\, d\varphi}{(2\pi)^3}\, \langle|\delta|^2\rangle_k\, e^{-ikr\sin\theta}$$

$$= \frac{4\pi}{(2\pi)^3} \int k^2 dk\, \langle|\delta|^2\rangle_k\, \frac{\sin kr}{kr}$$

Finally $\xi(r) = \int \frac{dk}{k} \left[\frac{k^3}{2\pi^2} \langle |\delta|^2 \rangle_k \right] \frac{\sin kr}{kr}$

It is conventional to call the quantity between brackets the (dimensionless) power spectrum of δ:

$$\boxed{\mathcal{P}_\delta(k) \equiv \frac{k^3}{2\pi^2} \langle |\delta(\vec{k})|^2 \rangle}$$

for whatever $\vec{k}$ of given modulus k

Then $\xi(r) = \int \frac{dk}{k} \mathcal{P}_\delta(k) \frac{\sin kr}{kr}$

and $\langle \delta(\vec{k}) \delta^*(\vec{k}') \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_\delta(k)\, \delta^{(3)}(\vec{k} - \vec{k}')$

- notion of primordial spectrum

The primordial spectrum of a perturbation $\delta(\vec{k}, t)$ is defined as the power spectrum at a time such that $\vec{k}$ is outside the Hubble radius $(|\vec{k}| \ll a(t)H(t))$:

$$\mathcal{P}_\delta(k) = \frac{k^3}{2\pi^2} \langle |\delta(\vec{k}, t)|^2 \rangle \quad \text{with } k \ll aH$$

(this definition makes sense because for the quantity of interest seen in this course, $\langle |\delta(\vec{k}, t)|^2 \rangle$ does not depend on t as long as $k \ll a(t) H(t)$: no evolution beyond R_H).

primordial spectrum of GWs in De Sitter

We define the primordial spectrum of GWs as

$$\mathcal{P}_h \equiv \frac{k^3}{2\pi^2} \sum_{ij} \left\langle \left| H^T_{ij}(\vec{k},t) \right|^2 \right\rangle \quad \text{for } k \ll a(t)H(t)$$

so, using our previous results:

$$\boxed{\mathcal{P}_h(k) = \frac{k^3}{2\pi^2} \frac{32\pi G}{k^3} H_i^2 = \frac{16}{\pi} G H_i^2 = \frac{16}{\pi} \left(\frac{H_i}{M_p}\right)^2}$$

Or, using the inflaton potential value V_i and $H_i^2 = \frac{8\pi G}{3} V_i$:

$$\boxed{\mathcal{P}_h(k) = \frac{2}{3}(8 G)^2 V_i = \frac{128}{3} \frac{V_i}{M_p^4}}$$

primordial spectrum and tilt in Quasi De Sitter

In an arbitrary model of inflation, the calculation of $\mathcal{P}_h(k)$ can be done by integrating $y'' + \left(k^2 - \frac{a''}{a}\right) y = 0$ in order to find the exact mode function corresponding to a given expansion law $a(\tau)$. Analytical results can be found by expanding at various order around the exact De Sitter case $a \propto \frac{1}{\tau}$, $H =$ constant

At first order, one can assume that when $\mathcal{P}_h(k)$ is computed for a given k, the calculation above will remain valid if we assume that H is constant in a neighbourhood of the time at which $k = aH$. Indeed, when $k \gg aH$, the equation giving the mode function is $g'' + k^2 g = 0$ and has a universal (properly normalized) solution $g = \frac{1}{\sqrt{2k}} e^{-ik\tau}$, not depending on $a(\tau)$. When $k \gg aH$, the mode function comes from $g'' - \frac{a''}{a} g = 0$ which gives $g \propto a$ and $h = \text{constant}$, even if $H(\tau)$ is not constant. So, the previous calculation was correct under the assumption that H is constant in a narrow range close to $k = a(\tau) H(\tau)$, not throughout inflation!

In this approximation, called the quasi De Sitter approximation, we then have:

$$\boxed{\mathcal{P}_h(k) = \frac{16G}{\pi} H_k^2 \quad \text{where } H_k \text{ means "}H(\tau)\text{ at the time } \tau \text{ when } k = a(\tau) H(\tau)\text{"}}$$

How to compute H_k concretely in a given inflationary model?

We remember that observable scales cross the Hubble radius roughly $N = \Delta N_{\text{post-inflation}}$ e-folds before the end of inflation. For a very precise computation of H_k one should assume a post-inflationary evolution (and hence a value for the energy density ρ_{end} at the end of inflation). For a rough calculation, we can remember that if inflation takes place e.g. at the GUT scale, then H_k should be evaluated ~ 60 e-folds before the end of inflation. Then, one should integrate the following relation:

$$N = \int_t^{t_{end}} \frac{da}{a} = \int_t^{t_{end}} H dt = \int_\varphi^{\varphi_{end}} \left(\frac{H dt}{d\varphi}\right) d\varphi = \int_\varphi^{\varphi_{end}} -\frac{3H^2}{V'} d\varphi = \int_\varphi^{\varphi_{end}} -\frac{8\pi G V}{V'} d\varphi$$

using $\varphi' \sim -\frac{V'}{3H}$ during slow roll

In summary, one should:

- compute first φ_{end} (usually given by breaking of SR conditions)
- find φ such that $\int_\varphi^{\varphi_{end}} -8\pi G \frac{V}{V'} d\varphi$ equals e.g. $N=60$
- compute $H^2 = \frac{8\pi G}{3} V$ for this value of φ: this is H_k

So, $\mathcal{P}_h(k)$ is a non-trivial function of k. At first-order, it can be parametrized as a power law with a tilt that we will now estimate.

Tensor tilt n_T:

Let us choose an arbitrary pivot scale k_* and write the tensor spectrum as:

$$\mathcal{P}_h(k) = \mathcal{P}_h(k_*)\left(\frac{k}{k_*}\right)^{n_T} \quad \text{with } n_T \equiv \left.\frac{d\ln \mathcal{P}_h}{d\ln k}\right|_{k_*}$$

We can compute n_T at first order in the slow-roll parameter. Let us write $d\ln\mathcal{P}_h/d\ln k$ as a finite difference:

$$n_T = \frac{\ln \mathcal{P}_h(k_*+dk) - \ln \mathcal{P}_h(k_*)}{\ln(k_*+dk) - \ln k_*}$$

Using $\mathcal{P}_h(k) \propto H_k^2$ we get

$$n_T = \frac{\ln\left(H_{(k_*+dk)}\right)^2 - \ln H_{k_*}^2}{\ln\left(1+\frac{dk}{k_*}\right)} \qquad \text{with } H_{k_*+dk} \equiv H_{k_*} + dH$$

$$= \frac{2\ln\left(1+\frac{dH}{H_*}\right)}{\ln\left(1+\frac{dk}{k_*}\right)} = 2\,\frac{dH}{H_*}\,\frac{k_*}{dk} \quad \text{at first order.}$$

By definition, $\begin{cases} k_* \text{ is the scale such that } k_* = a_* H_* \\ k_*+dk \text{ " " " " } (k_*+dk) = (a_*+da)(H_*+dH) \end{cases}$

The second relation gives:

$$k_*+dk = \left(a_* + \frac{da}{dt}\frac{dt}{dH}dH\right)(H_*+dH) = \left(a_* + \frac{a_* H_*}{\dot{H}_*}dH\right)(H_*+dH)$$

$$= a_* H_* + a_*\left(1+\frac{H_*^2}{\dot{H}_*}\right)dH + \mathcal{O}(dH^2) \qquad \left(\dot{} \equiv \frac{d}{dt} \text{ proper time}\right)$$

The first slow-roll condition gives $1 \ll \left|\frac{H_*^2}{\dot{H}_*}\right|$ so:

$$k_* + dk = \underbrace{a_* H_*}_{\rotatebox{90}{=}\ k_*} + a_* \frac{H_*^2}{\dot{H}_*} dH$$

$$\Leftrightarrow \frac{dH}{dk} = \frac{\dot{H}_*}{H_*^2} a_*^{-1}$$

So $$n_T = 2 \underbrace{\frac{k_*}{H_*} a_*^{-1}}_{=1} \frac{\dot{H}_*}{H_*^2} = 2 \frac{\dot{H}_*}{H_*^2}$$

The tensor tilt is given by $2\frac{\dot{H}}{H^2}$ evaluated when the pivot scale k_* crosses the Hubble radius ($k_* = a_* H_*$)!

Note that this quantity is related to the first slow-roll parameter of Liddle & Lyth:

$$\varepsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \simeq \frac{(-3H\dot{\varphi})^2}{16\pi G \left(\frac{3H^2}{8\pi G}\right)^2} = \frac{4\pi G\, \dot{\varphi}^2}{H^2} = -\frac{\dot{H}}{H^2}$$

So $n_T = -2\varepsilon_*$ ($\varepsilon_* = \varepsilon$ evaluated when pivot scale crosses Hubble radius)

In summary, in the quasi De Sitter approximation:

$$\boxed{\begin{aligned} &\mathcal{P}_h(k) = A_T \left(\frac{k}{k_*}\right)^{n_T} \quad \text{with:} \\ &A_T = \frac{16\,G}{\pi} H_*^2 = \frac{16}{\pi}\left(\frac{H_*}{M_P}\right)^2 = \frac{128}{3} \frac{V_*}{M_P^4} \\ &n_T = -2\varepsilon_* \leq 0 \end{aligned}}$$