III.5. Scalar perturbations:

In the Newtonian gauge, there are two scalar perturbations of the metric: Φ and Ψ, coupled with the perturbations of the inflaton field: $\delta\varphi$.

Einstein equations $\Rightarrow \delta G^j_i = 8\pi G\, \delta T^j_i$

$$\Rightarrow \partial_i \partial^j (\Phi - \Psi) = 8\pi G \partial_i(\delta\varphi)\, \partial^j(\delta\varphi)$$

The RHS is of order 2 in perturbations. So, at the level of linear perturbation theory, $\underline{\Phi = \Psi}$ during inflation.

The evolution of $\delta\varphi$, Φ is governed by two equations (one eq. of motion (2[nd] order dif. eq.) + one constraint eq. (1[st] order dif. eq.)).
For instance:

Perturbed Klein-Gordon equation: (in Fourier space)

$$\delta\ddot{\varphi}_{\vec{k}} + 3H\,\delta\dot{\varphi}_{\vec{k}} + \left[\frac{k^2}{a^2} + \frac{\partial^2 V}{\partial\varphi^2}(\bar{\varphi})\right]\delta\varphi_{\vec{k}} = 4\dot{\bar{\varphi}}\,\dot{\Phi}_{\vec{k}} - 2\frac{\partial V}{\partial\varphi}(\bar{\varphi})\Phi_{\vec{k}}$$

Einstein equation $\delta G^0_i = 8\pi G\, \delta T^0_i$:

$$\dot{\Phi}_{\vec{k}} + H\Phi_{\vec{k}} = 4\pi G\, \dot{\bar{\varphi}}\, \delta\varphi_{\vec{k}}$$

[The 2[nd] equation obviously admits a trivial solution that we will not consider because it decays: $\Phi_k \propto \frac{1}{a}$, $\delta\varphi_k = 0$]

The action reads: $S = \int d^4x \sqrt{|g|}\left[\frac{R}{16\pi G} + \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)\right]$

Mukhanov variable

Since $\delta\varphi$ and Φ are coupled to each other, but contain only one propagating degree of freedom (only one equation of motion!), it should be possible to find a quantity ξ (= gauge-invariant linear combination of $\delta\varphi$ and metric pert.) with a canonical action:

$$S = \int d^4x \left[\frac{1}{2}\left(\xi'^2 - (\partial_i \xi)^2 \right) - V_\xi(\xi) + \text{divergence terms} \right.$$
$$+ \text{vector sector} + \text{tensor sector}$$
$$\left. + \text{terms negligible for linear perturb. theory} \right]$$

This ξ was first introduced by Mukhanov. In the Newtonian gauge it reduces to:

$$\xi = a\,\delta\varphi + \frac{\varphi'}{H}\phi = a\left(\delta\varphi + \frac{\dot{\varphi}}{H}\phi\right) \qquad \left({}' = \frac{\partial}{\partial \tau} \text{ conformal time} \qquad \dot{} = \frac{\partial}{\partial t} \text{ proper time} \right)$$

The effective potential is surprisingly simple:

$$V_\xi = \frac{1}{2}\frac{z''}{z}\xi^2, \text{ with } z = \frac{\varphi'}{H} = a\frac{\dot{\varphi}}{H} \left(= \frac{a}{\sqrt{4\pi G}}\sqrt{2 - \frac{a a''}{a'^2}} \text{ using } \dot{H} = -4\pi G \dot{\varphi}^2 \right)$$

The equation of motion is then (in Fourier space):

$$\xi_k'' + \left(k^2 - \frac{z''}{z}\right)\xi_k = 0$$

<u>Remark:</u> the potential $V(\varphi)$ is absent from this equation, because it has been eliminated using the KG equation. But it plays a role, since it dictates the background evolution. Actually, using KG, one can show that:

$$\frac{z''}{z} = \left(\frac{\varphi'}{H}\right)'' \frac{H}{\varphi'} = -a^2 \left\{ \frac{\partial^2 V}{\partial \varphi^2} + 8\pi G \frac{\varphi'}{aH}\frac{\partial V}{\partial \varphi} + H\left(\frac{H'}{a^2 H^2}\right)' - 2H^2 \right\}$$

Finally, note that this simple equation of motion was obtained assuming that $\bar{\varphi}$ drives the background evolution: $H^2 = \frac{8\pi G}{3}\rho$ with $\rho = \frac{1}{2}\dot{\bar{\varphi}}^2 + V(\bar{\varphi})$.

De Sitter limit:

If H=constant, we must take V=constant, $\varphi' = H' = 0$. In this case, $z = 0$ but $\frac{z''}{z}$ remains finite: $\frac{z''}{z} = 2a^2H^2$.

In this limit: $\xi = a\delta\varphi + z\Phi = a\delta\varphi$.
The scalar field decouples from scalar metric perturbations. It is possible to quantize ξ and $\delta\varphi$, but these fluctuations do not generate any metric fluctuations. This limit is at odds with observations (need for metric fluctuations in early universe).
So, the first-order sensible calculation must be performed in the Quasi De Sitter approximation.

Quasi De Sitter approximation

We assume that in the vicinity of the time at which $k = aH$, $\mathring{\varphi}$ and H are constant and equal to $\{\mathring{\varphi}_K, H_K\}$ (which are $\mathring{\varphi}|_{k=aH}$ and $H|_{k=aH}$).
In this approximation, we can take $z \propto a$ and $\frac{z''}{z} \simeq \frac{a''}{a}$. Then:

$$\xi_{\vec{k}}'' + \left(k^2 - \frac{a''}{a}\right)\xi_{\vec{k}} = 0 \quad \text{and} \quad \frac{a''}{a} = \frac{2}{\tau^2} = 2a^2H_K^2.$$

Since $\xi_{\vec{k}}$ has a canonical action, $W = \xi_{\vec{k}}\xi_{\vec{k}}^{*\prime} - \xi_{\vec{k}}'\xi_{\vec{k}}^{*} = i$ and the mode function in the limit $k \gg aH \sim \frac{a''}{a}$ should read:

$$\xi_{\vec{k}} = \frac{1}{\sqrt{2k}}e^{-ik\tau} \qquad (\text{limit } k \gg aH \Leftrightarrow k\tau \gg 1)$$

The mode function at all times is found in analogy with the tensor case:

$$\xi_k^{\mathcal{P}} = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{kz}\right) e^{-ikz}$$

Outside the Hubble radius it tends to:

$$\xi_k^{\mathcal{P}} \xrightarrow[(\Leftrightarrow kz \ll 1)]{k \ll aH} -\frac{1}{\sqrt{2k}} \frac{i}{kz} = \frac{i a H_k}{\sqrt{2k^3}}$$

The primordial spectrum for $\xi_k^{\mathcal{P}}$ would be given by $\langle |\xi_k^{\mathcal{P}}|^2 \rangle = + \frac{a^2 H_k^2}{2k^3}$, but this is not useful. We want to compute the primordial spectrum for a quantity which remains constant as long as $kz \ll 1$, at all stages: inflation, RD, MD, etc..

Curvature perturbation:

It will be shown in IV.3 that outside the Hubble radius, the curvature perturbation $\mathcal{R}$ (which reads $\mathcal{R} = \Psi - \frac{1}{3}\frac{\delta \rho_{tot}}{(\rho + P)_{tot}}$ in the Newtonian gauge) is conserved on super-Hubble scales during all stages under very generic conditions (which are always fulfilled during and after single-field inflation). So, we whish to compute the primordial spectrum $\mathcal{P}_{\mathcal{R}} = \frac{1}{2\pi^2} k^3 \langle |\mathcal{R}_k^{\mathcal{P}}|^2 \rangle$. For this purpose, we need to relate $\mathcal{R}_k^{\mathcal{P}}$ to $\xi_k^{\mathcal{P}}$ at the time when k has just crossed the horizon: $k \ll aH$ ($kz \ll 1$), but $\{H, \varphi'\}$ are still equal to $\{H_k, \varphi'_k\}$.

Using $T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu}\left(\frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right)$ we compute:

$$\ast \quad \bar\rho + \bar p = \varphi'^2$$

$$\ast \quad \delta\rho = \delta T^0_0 = a^{-2}\varphi'\delta\varphi' + \frac{\partial V}{\partial \varphi}\delta\varphi - a^{-2}\varphi'^2\phi$$

The expression of $\delta\rho$ can be simplified by making use of the (0i) Einstein equation:

$$\dot\phi + H\phi = 4\pi G\dot\varphi\delta\varphi \Leftrightarrow \phi' + aH\phi = 4\pi G\varphi'\delta\varphi$$

as well as the (00) Einstein equation:

$$\delta G^0_0 = 2a^{-2}\left\{-3\left(\frac{a'}{a}\right)^2\phi - 3\frac{a'}{a}\psi' - k^2\psi\right\} = 8\pi G\delta T^0_0 = 8\pi G\delta\rho$$

with $\phi = \psi$. This gives:

$$\delta\rho = \frac{1}{4\pi G}\left[-3\frac{a'}{a}(\phi' + aH\phi) - k^2\phi\right]$$
$$= \frac{1}{4\pi G}\left[-3\frac{a'}{a}\,4\pi G\varphi'\delta\varphi - k^2\phi\right]$$

So, an alternative expression for $\delta\rho$ is:

$$\delta\rho = -3aH\varphi'\delta\varphi - \frac{k^2}{4\pi G}\phi$$

Then, $\mathcal{R} = \phi + \frac{1}{3\varphi'^2}\left[3aH\varphi'\delta\varphi + \frac{k^2}{4\pi G}\phi\right]$

Using $\dot H = -4\pi G\dot\varphi^2 \Leftrightarrow aH' = -4\pi G\varphi'^2$:

$$\mathcal{R} = \frac{aH}{\varphi'}\delta\varphi + \left[1 - \frac{k^2}{aH'}\right]\phi$$

In the long wave-length limit, for modes with $\left|\frac{k^2}{aH'}\right| \ll 1$,

$$\mathcal{R} \longrightarrow \frac{aH}{\varphi'}\delta\varphi + \phi = \frac{H}{\varphi'}\left(a\delta\varphi + \frac{\varphi'}{H}\phi\right) = \frac{H}{\varphi'}\xi = \frac{\xi}{z}$$

So: $\langle|\mathcal{R}_{\vec k}|^2\rangle = \left\langle\left|\frac{\xi_{\vec k}}{z}\right|^2\right\rangle = \frac{H_k^2}{(a\dot\varphi_k)^2}\langle|\xi_{\vec k}|^2\rangle = \frac{H_k^4}{\dot\varphi_k^2\, 2k^3}$

Finally:

$$\boxed{\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{2\pi^2} k^3 \frac{\langle |\xi_k^{\mathcal{R}}|^2 \rangle}{z^2} = \frac{1}{4\pi^2} \frac{H_k^4}{\dot{\varphi}_k^2}}$$

Using the slow-roll expressions, we can write various equivalent expressions:

$$\boxed{\mathcal{P}_{\mathcal{R}}(k) = -\frac{G}{\pi} \frac{H_k^4}{H'_k} = \frac{128\pi}{3} \frac{V_k^3}{M_P^6 V_k'^2} = \frac{8}{3} \frac{V_k}{M_P^4 \varepsilon_k}}$$

We can immediately infer the scalar-to-tensor ratio:

$$\boxed{r \equiv \frac{\mathcal{P}_h(k)}{\mathcal{P}_{\mathcal{R}}(k)} = \frac{\frac{128}{3}\frac{V_k}{M_P^4}}{\frac{8}{3}\frac{V_k}{M_P^4 \varepsilon_k}} = 16\,\varepsilon_k}$$

The dependence of V_k, ε_k, H_k on k implies that $\mathcal{P}_{\mathcal{R}}$, $\mathcal{P}_h$ and r slighly scale-dependent.

Scalar tilt:

At first order, we can write $\mathcal{P}_{\mathcal{R}}(k) = A_S \left(\frac{k}{k_*}\right)^{n_S - 1}$

with k_* = arbitrary pivot scale, and $n_S \equiv \frac{d\ln \mathcal{P}_{\mathcal{R}}}{d\ln k} - 1$

(note the difference with $n_T = \frac{d\ln \mathcal{P}_h}{d\ln k}$: this is just a matter of conventions).

We can relate n_S to the slow-roll parameters, as we did for n_T:

$$n_S - 1 = \frac{\ln \mathcal{P}_{\mathcal{R}}(k_* + dk) - \ln \mathcal{P}_{\mathcal{R}}(k_*)}{\ln(k_* + dk) - \ln(k_*)}$$

$$= \frac{\ln\left[\frac{(H_* + dH)^4}{(\dot{\varphi}_* + d\dot{\varphi})^2}\right] - \ln\left[\frac{H_*^4}{\dot{\varphi}_*^2}\right]}{\frac{dk}{k_*}}$$

So:

$$n_S - 1 = \frac{4\left[\ln(H_* + dH) - \ln H_*\right] - 2\left[\ln(\dot{\varphi}_* + d\dot{\varphi}) - \ln \dot{\varphi}_*\right]}{\frac{dk}{k_*}}$$

$$= 4\,\frac{k_*}{H_*}\,\frac{dH}{dk} - 2\,\frac{k_*}{\dot{\varphi}_*}\,\frac{d\dot{\varphi}}{dk}$$

In the tensor section, we have shown that $dk = \frac{a_* H_*^2}{\dot{H}_*}\,dH$. So, the first term is:

$$4\,\frac{k_*}{H_*}\,\frac{dH}{dk} = 4\,\frac{k_*\,\dot{H}_*}{a_*\,H_*^3} \underset{\text{since } k_* = a_* H_*}{=} 4\,\frac{\dot{H}_*}{H_*^2} = -4\,\varepsilon_*$$

since $k_* = a_* H_*$

first slow-roll parameter, evaluated when $k_* = aH$

The second term reads:

$$-2\,\frac{k_*}{\dot{\varphi}_*}\,\frac{d\dot{\varphi}}{dt}\,\frac{dt}{dH}\,\frac{dH}{dk} = -2\,\frac{k_*}{\dot{\varphi}_*}\,\frac{\ddot{\varphi}_*}{\dot{H}_*}\,\frac{\dot{H}_*}{a_* H_*^2} = -2\,\frac{\ddot{\varphi}_*}{\dot{\varphi}_* H_*}$$

The S-R relation $\dot{\varphi} = -\frac{V'}{3H}$ implies:

$$\ddot{\varphi} = -\frac{V''\dot{\varphi}}{3H} + \frac{V'\dot{H}}{3H^2} = -\frac{V''\dot{\varphi}}{3H} - \frac{\dot{\varphi}\dot{H}}{H}$$

So the second term reads:

$$\frac{2}{\dot{\varphi}_* H_*}\left(\frac{V''_*\,\dot{\varphi}_*}{3H_*} + \frac{\dot{\varphi}_*\dot{H}_*}{H_*}\right) = 2\,\frac{V''_*}{3H_*^2} + 2\,\frac{\dot{H}_*}{H_*^2} = \frac{2V''_*}{8\pi G V_*} + 2\,\frac{\dot{H}_*}{H_*^2}$$

$$= 2\,\eta_* - 2\,\varepsilon_*$$

Finally: $n_S - 1 = -4\,\varepsilon_* + 2\eta_* - 2\,\varepsilon_* = -6\,\varepsilon_* + 2\eta_*$

$$\boxed{n_S = -6\,\varepsilon_* + 2\eta_* + 1}$$

1st and 2nd SR parameters

Summary of primordial spectra:

- **Scalars:** $\mathcal{R}$ = curvature perturbation

$$\mathcal{P}_{\mathcal{R}} = \frac{1}{2\pi^2} k^3 \langle |\mathcal{R}_k|^2 \rangle = A_S \left(\frac{k}{k_*}\right)^{n_S - 1}$$

with
$$\begin{cases} A_S \simeq \frac{1}{4\pi^2} \frac{H_*^4}{\dot{\varphi}_*^2} = -\frac{H_*^4}{\pi M_p^2 H_*'} = \frac{128\pi}{3} \frac{V_*^3}{M_p^6 V_*'^2} = \frac{8}{3} \frac{V_*}{M_p^4 \varepsilon_*} \\ n_S \simeq -6\varepsilon_* + 2\eta_* + 1 \end{cases} \qquad (M_p^{-2} = G)$$

with the SR parameters
$$\begin{cases} \varepsilon = \frac{1}{16\pi G} \left(\frac{V'}{V}\right)^2 = -\frac{\dot{H}}{H^2} \\ \eta = \frac{1}{8\pi G} \frac{V''}{V} \end{cases}$$

- **Tensors** h_λ = each degree of polarization of GW, combined in the total power spectrum

$$\mathcal{P}_h(k) = \frac{k^3}{2\pi^2} \sum_{i,j} \langle |H_{ij}|^2 \rangle = \frac{k^3}{4\pi^2} \left(\langle |h_1|^2 \rangle + \langle |h_2|^2 \rangle \right) = A_T \left(\frac{k}{k_*}\right)^{n_T}$$

with
$$\begin{cases} A_T \simeq \frac{16}{\pi} \frac{H_*^2}{M_p^2} = \frac{128}{3} \frac{V_*}{M_p^4} \\ n_T \simeq -2\varepsilon_* \end{cases}$$

- **Ratio** $r = \mathcal{P}_h(k_*) / \mathcal{P}_{\mathcal{R}}(k_*) = A_T / A_S = 16\,\varepsilon_*$

- **Self-consistency relation** $r = -8\, n_T$