IV.2. A stochastic theory

IC (Initial conditions) for perturbations such that **initial value** of perturbation "δ_i" for mode $\vec{k}$ obeys some distribution of probability. The properties of IC's can be probed by observations (which kind of probability? Which variance and higher moments? Which correlation between different δ_i's for different components?).

In order to make theoretical predictions, it is convenient to write the IC as stochastic numbers:

$$\delta_i(\vec{k}, t_{ini}) = C(\vec{k})$$

C ← stochastic number with some statistical properties: maybe gaussian, maybe not, ... ; maybe isotropic, " " , ...

Then one can compute observables and see if they agree with observations or not.

Inflation picks up one particular possibility for these stochastic IC's which turns out to be in agreement with all current observations.

Let us charcterize more precisely these standard IC's.

We have to solve the equation of motion of N fluids (N 2nd order linear, coupled differential equations). So there must be 2N independent solutions $\alpha = 1, 2, \dots, 2N$.
Since the system is linear, the general solution for the perturbation δ_i of wave vector $\vec{k}$ at time τ reads:

$$\forall i = 1, \dots N \qquad \delta_i(\vec{k}, \tau) = \sum_{\alpha=1}^{2N} C_\alpha(\vec{k})\, f_i^\alpha(k, \tau)$$

Note that f_i^α depends on k and not $\vec{k}$ because the equations preserve the isotropy of FLRW (coefficients depend on k, not on each k^i).

⤳ since IC's are stochastic, we can consider $C_\alpha(\vec{k})$ as a stochastic number

⤳ isotropy of FLRW universe $\Rightarrow$ statistical properties of $C_\alpha(\vec{k})$'s depend only on k

⤳ both observations and theoretical predictions of inflation indicate that $C_\alpha(\vec{k})$'s are gaussian: $\mathcal{P}(C_\alpha(\vec{k})) = \mathcal{N}_k \exp\left[-\frac{|C_\alpha(\vec{k})|^2}{2\,\sigma_\alpha(k)^2}\right]$

(see chapter on inflation: $\mathcal{P} = |\psi|^2$ where ψ is a gaussian wave functional, initially describing vacuum)

⇝ both observations and theoretical predictions of (simplest category of) inflationary models indicate that all C_α's but one are zero, or vanishingly small, or just irrelevant for observables quantity. The relevant solution (say, with $\alpha=1$) is the "growing adiabatic mode". This will be justified in the next sections. As a result:

$$\forall i=1,\dots N \qquad \delta_i(\vec{k},\tau) = C_1(\vec{k})\, f^1_i(k,\tau)$$

gaussian stochastic number with some variance $\sigma_1(k)$ and some power spectrum $\mathcal{S}_{C_1}(k) = \frac{1}{2\pi^2} k^3 \sigma_1^2(k)$

Note that if this is true, all δ_i's are correlated (single stochastic number $C_1(\vec{k})$ for all perturbations $\delta_i(\vec{k},\tau)$, $i=1\dots N$)

So, in order to compute the spectrum today (e.g. $\mathcal{S}_{\delta_1}(k) = \frac{1}{2\pi^2} k^3 \langle |\delta_1(\vec{k},\tau)|^2 \rangle$) one can follow one of the two equivalent approaches below:

ⓐ * Normalize all δ_i's initially to one standard deviation: $\delta_i(k,\tau_{ini}) = \langle |C_1(\vec{k})|^2\rangle^{1/2} f_i^1(k,$
* integrate the system for each k (not each $\vec{k}$!)
* at any time τ, the values $\delta_i(k,\tau)$ can be interpreted as the <u>variance</u> of the actual random $\delta_i(\vec{k},\tau)$'s
* at the end $\frac{1}{2\pi^2}k^3\,\delta_i^2(k,\tau)$ does represent the power spectrum today

ⓑ * Normalize all C_1's initially to unity: $C_1^{(\vec{k})} \equiv 1$
* integrate the system for each k
* at the end, multiply the squared perturbations by the primordial spectrum for C_1:

$$\mathcal{S}_{\delta_i}(k) = \frac{1}{2\pi^2}k^3\,\delta_i^2(k,\tau)\langle |C_1(\vec{k})|^2\rangle = \delta_i^2(k,\tau)\,\mathcal{S}_{C_1}(k)$$

The two approaches are equivalent because the system is linear; hence, the time-evolution and the initial conditions are separable:

$$\frac{k^3}{2\pi^2}\langle |\delta_i(\vec{k},\tau)|^2\rangle = \underbrace{\frac{k^3}{2\pi^2}\left\langle |f_i^1(k,\tau_{ini})\,C_1(\vec{k})|^2\right\rangle}_{\substack{\parallel\\ \text{primordial spectrum}\\ \text{for } \delta_i}}\underbrace{\left(\frac{f_i^1(k,\tau)}{f_i^1(k,\tau_{ini})}\right)^2}_{\substack{\parallel\\ \text{transfer function}\\ \text{for } \delta_i}}$$

<u>Remark:</u> in general, in some complicated cosmological models, two solutions C_1, C_2 or more can be important; then, observables depend on power spectrum of each C_α plus on a possible cross-correlation:

E.g. $\delta_i(\vec{k},\tau) = C_1(\vec{k})\, f_i^1(k,\tau) + C_2(\vec{k})\, f_i^2(k,\tau)$

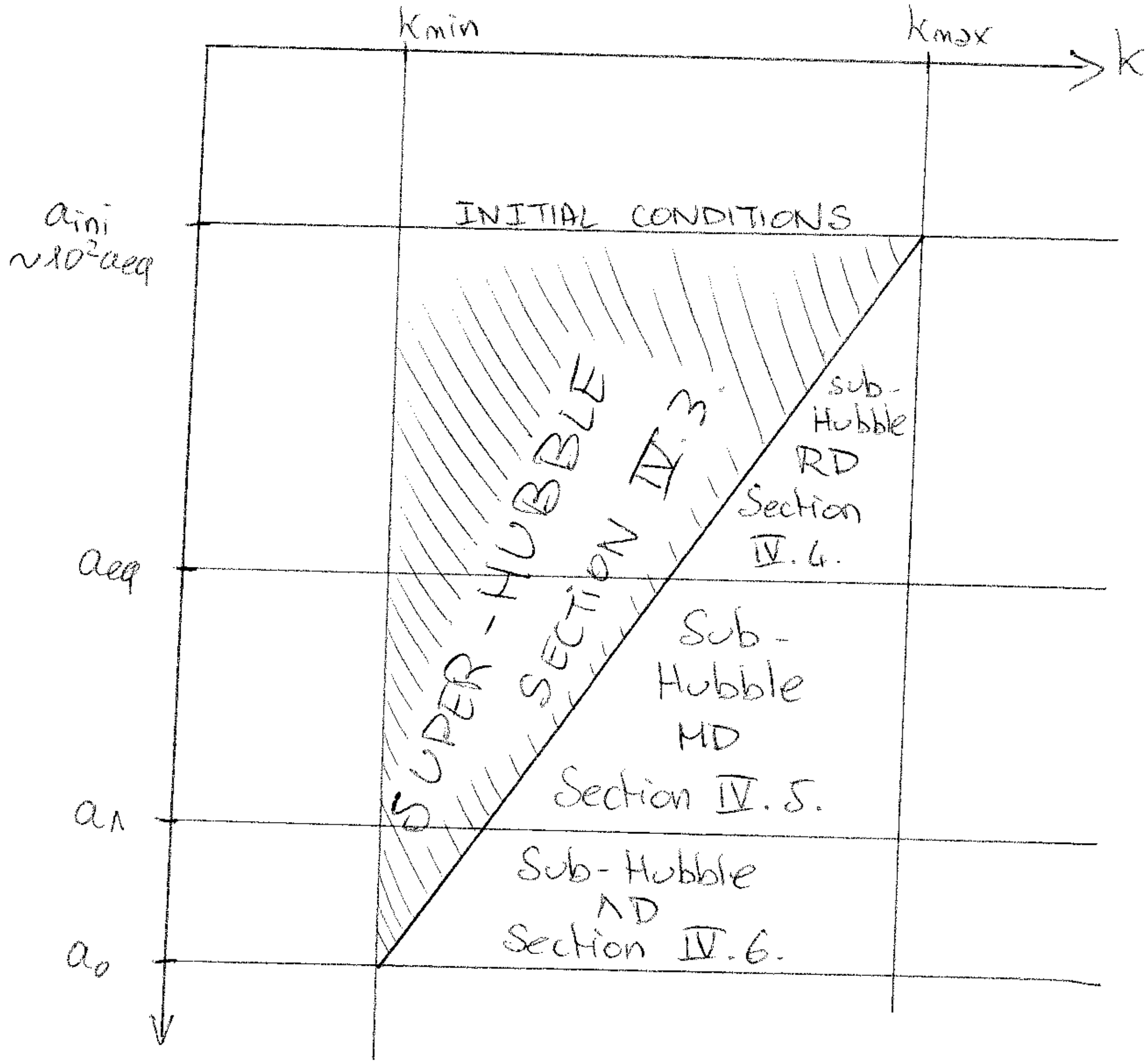
$$\Rightarrow \mathcal{S}_{\delta_i}(k,\tau_0) = \mathcal{S}_{C_1}(k)\left(f_i^1(k,\tau)\right)^2 + \mathcal{S}_{C_2}(k)\left(f_i^2(k,\tau)\right)^2 + \mathcal{S}_{C_1,C_2}(k)\left(f_i^1(k,\tau)\, f_i^2(k,\tau)\right)$$

where the cross-correlation power spectrum $\mathcal{S}_{C_1,C_2}(k)$ comes from a possible correlation between $C_1(\vec{k})$ and $C_2(\vec{k})$:

$$\mathcal{S}_{C_1,C_2}(k) = \left[\mathcal{S}_{C_1}(k)\,\mathcal{S}_{C_2}(k)\right]^{1/2} \cos\Theta(k)$$

correlation angle
($\Theta=0$ for statistically inde-
-pendent C_1 and C_2)

In the next sections we will follow the evolution of modes in the different regimes:



Note that what we will call "Initial conditions" means "characterization of perturbations when all relevant modes are outside the Hubble radius", $k_{max} \lesssim aH$. Hence what we call "initial conditions" in this chapter is related to the final result of Chapter III (inflation)...