IV.3. Super-Hubble evolution:

IV.3.A. Adiabatic Initial Conditions (I.C.)

Here we want to prove that the most natural set of ICs for the simplest cosmological models read:

(1) All δ_i's are related to each other through:

$$\forall i,j \qquad \frac{\delta\rho_i}{\bar{\rho}_i+\bar{p}_i} = \frac{\delta\rho_j}{\bar{\rho}_j+\bar{p}_j} \iff \frac{\delta_i}{1+w_i} = \frac{\delta_j}{1+w_j}$$

(2) The curvature perturbation (defined in Chapter III),

$\mathcal{R} \equiv \psi - \frac{1}{3}\frac{\delta\rho_T}{\rho_T+p_T}$, remains constant for $k \ll aH$ (hence, it does not change for a given mode k between $k \sim aH$ during inflation and the re-entry during R/M/Λ domination)

(Note: $\mathcal{R} = \psi - \frac{1}{3}\frac{\delta\rho_T}{\rho_T+p_T}$ "total" holds in the newtonian gauge.

Indeed, in general, one can construct a gauge-invariant quantity which reduces:
- * to $\psi - \frac{1}{3}\frac{\delta\rho_T}{\rho_T+p_T}$ in Newtonian gauge
- * to the perturbation of the spatial curvature in the "comoving gauge", i.e. the gauge where $\delta\rho_T \equiv 0$)

For each species, we can define $\xi_i = \varphi - \frac{1}{3}\frac{\delta\rho_i}{\rho_i + p_i}$.
If we can prove that:

$$\begin{cases} \forall i,j & \xi_i = \xi_j \\ \forall i & \dot{\xi}_i = 0 \quad \text{for } k \ll aH \end{cases}$$

then the conditions ① and ② are satisfied. Indeed:

$\rightsquigarrow$ $\xi_i = \xi_j \Rightarrow \frac{\delta\rho_i}{\rho_i + p_i} = \frac{\delta\rho_j}{\rho_j + p_j}$ (condition ①)

$\rightsquigarrow$ $\mathcal{R} = \sum_i (\bar{\rho}_i + \bar{p}_i)\xi_i \,/\, \sum_i (\bar{\rho}_i + \bar{p}_i)$

If all ξ_i's are equal to each other, this gives

$\mathcal{R} = \xi_i$ and $\dot{\xi}_i = 0 \Rightarrow \dot{\mathcal{R}} = 0$ (condition ②)

Why do minimal cosmological models predict $\forall i,j \;\xi_i = \xi_j$?

Outside the Hubble radius, it is natural to have all ξ_i's equal to each other for various reasons:

(i) THERMAL EQUILIBRIUM.

Suppose that at a given time all species are in thermal equilibrium with common temperature T.

$\rightsquigarrow$ For relativistic species: $n_i = (\dots)T^3$, $\rho_i = (\dots)T^4$, $p_i = \frac{1}{3}\rho_i$

So $\frac{\delta\rho_i}{\rho_i + p_i} = \frac{3}{4}\frac{\delta\rho_i}{\rho_i} = \frac{3}{4}\left(4\frac{\delta T}{T}\right) = 3\frac{\delta T}{T} = \frac{\delta n_i}{n_i}$ (assuming no chemical potentials)

$\rightsquigarrow$ For non-relativistic species: $n_i = (\dots)T^3$, $\rho_i = m_i n_i$, $p_i \ll \rho_i$

So $\frac{\delta\rho_i}{\rho_i + p_i} = \frac{\delta\rho_i}{\rho_i} = \frac{\delta n_i}{n_i} = 3\frac{\delta T}{T}$ (assuming no chemical potentials)

The fact that $3\frac{\delta T}{T}$ is a unique function of $(\vec{x},t)$ ensures that $\forall i,j$ $\quad \frac{\delta\rho_i}{\bar{\rho}_i+\bar{p}_i} = \frac{\delta\rho_j}{\bar{\rho}_j+\bar{p}_j} \Longleftrightarrow \frac{\delta n_i}{\bar{n}_i} = \frac{\delta n_j}{\bar{n}_j}$

Hence, if at some time all species are in thermal equilibrium with $\mu_i=0$, they all share the same ξ_i (argument valid <u>inside</u> and <u>outside</u> R_H). In addition, we will prove later on that the entropy perturbation $S_{ij}=\xi_i-\xi_j$ <u>must remain null</u> as long as $k \ll aH$: so, even when some species decouple from the thermal bath, they keep this common value of ξ_i until re-entry inside R_H.

<u>Caveat</u>: this argument does not hold if some species carry a significant chemical potential (could occur for neutrinos, although there are strong limits from BBN) OR if some species were never in thermal equilibrium (e.g. dark matter particles; although supersymmetric candidates are <u>usually</u> thought to be in equilibrium at high temperature).

(ii) CREATION OF PARTICLES FROM SINGLE FIELD

In single-field inflation models, all particles are created through inflaton decay during reheating / preheating.

Let us consider a toy model with one inflaton φ and two matter fields X_1 and X_2 created e.g. through: $\varphi \longrightarrow X_1 + 2X_2$

It is clear that after the decay, $n_{X_2} \neq n_{X_1}$ (since $n_{X_2} = 2n_{X_1}$). Also, it is clear that n_{X_i} is a function of space since φ carries perturbations $\delta\varphi(\vec{x},t)$. However, the relative number perturbations $\frac{\delta n_{X_1}}{n_{X_1}} = \frac{\delta n_{X_2}}{n_{X_2}}$ are equal (by differentiation of $n_{X_2}(x) = 2n_{X_1}(x)$) and given by the same unique function of $\delta\varphi(x)$.

In general,

$$\left\{\begin{array}{c}\text{DECAY FROM}\\ \text{UNIQUE}\\ \text{INFLATON}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{UNIQUE}\\ \text{FUNCTION}\\ \text{OF SPACE}\\ \delta\varphi(x,t)\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{UNIQUE DENSITY}\\ \text{CONTRASTS}\\ \frac{\delta n_i}{n_i} = \frac{\delta\rho_i}{\bar{\rho}_i + \bar{p}_i}\end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{c}\text{UNIQUE}\\ \xi_i = \psi - \frac{1}{3}\frac{\delta\rho_i}{\bar{\rho}_i + \bar{p}_i}\end{array}\right.$$

We will prove later that once $S_{ij} \equiv \xi_i - \xi_j$ are zero, they cannot depart from zero as long as $k \ll aH$.

Caveat: does not hold if two or more fields decay

When all ξ_i's are equal to each other, all entropy perturbations $S_{ij} = \frac{\delta e_i}{\bar{e}_i + \bar{p}_i} - \frac{\delta e_j}{\bar{e}_j + \bar{p}_j} = \frac{\delta n_i}{\bar{n}_i} - \frac{\delta n_j}{\bar{n}_j}$ vanish. Hence, such initial conditions are called ADIABATIC INITIAL CONDITIONS.

<u>If IC's are adiabatic, why does $\xi_i = \xi_j$ remains true as long as $k \ll aH$, whatever happens?</u>

Let's assume that at some time t_i and for a mode with $k \ll aH$, one has: $\forall i,j \; \xi_i = \xi_j$ (because of (i) and/or (ii) above). Let us assume that this will remain true at least until re-entry inside the Hubble radius, even in case of complicated interactions between species.

Energy conservation follows from $D_\mu T^{(i)\mu}_{0} = 0$ or some function $Q^{(i)}(T^{(j)\mu}_{\nu})$ in case of coupling with other species. At the background level, this gives:

$$\dot{\bar{\rho}}_i + 3H(\bar{\rho}_i + \bar{p}_i) = Q^{(i)}(\bar{\rho}_j, ..)$$

and $Q^{(i)} = 0 \Rightarrow \bar{n}_i a^3 = \text{cte}$. The relation with $Q^{(i)} = 0$ is satisfied for species which are decoupled, or for which $n_i a^3 = \text{cte}$ is enforced by charge conservation, or thermal equilibrium with $Ta = \text{cte}$.

At the level of perturbations, $D_\mu T^{(i)\mu}_0 = Q^{(i)}(x^\mu)$ gives:

Newtonian gauge:
$$\left(\psi - \frac{1}{3}\frac{\delta\rho_i}{\bar\rho_i+\bar p_i}\right)' + \frac{\sum_i \partial_i T_i^0}{\bar\rho_i+\bar p_i} = -\frac{a'}{a}\frac{1}{\bar\rho_i+\bar p_i}\left(\frac{\bar p_i'}{\bar\rho_i'}\delta\rho_i - \delta p_i\right)$$
$$+ (\cdots)\left(\delta Q_i - \frac{\bar Q_i'}{\bar\rho_i'}\delta\rho_i\right)$$

$' = \frac{d}{d\tau}$ (conformal time)

(for details see e.g. astro-ph/0411703)

The term $\sum_i \partial_i T_i^0$ is what we called previously Θ_i.
It can be shown that for $k \ll aH$, this velocity term becomes completely negligible (in general, gradient/divergence terms are suppressed in this limit).

If the terms on the right-hand side vanish, then we are left with: $(\xi_i)' = 0$. Then, the entropy perturbation $S_{ij} = \xi_i - \xi_j$ is conserved for all pairs i,j and adiabatic initial conditions remain adiabatic (since S_{ij} cannot vary outside the Hubble radius). When the terms on the right-hand side are non-zero, this is not true anymore. Hence,

$\delta p_i - \frac{\bar p_i'}{\bar\rho_i'}\delta\rho_i$ is called the non-adiabatic pressure,
$\delta Q_i - \frac{\bar Q_i'}{\bar\rho_i'}\delta\rho_i$ " " " " " coupling.

However, the non-adiabatic terms cancel when p_i and Q_i can be written as arbitrary functions of ρ_i:
$$p_i(\vec x,t) = f(\rho_i(\vec x,t)), \quad Q_i(x,t) = g(\rho_i(\vec x,t)).$$

Indeed, this implies:

$$\begin{cases} \delta p_i = \dfrac{\partial f(\rho_i)}{\partial \rho_i}\,\delta\rho_i = \dfrac{\partial \bar{p}_i}{\partial \bar{\rho}_i}\,\delta\rho_i = \dfrac{\bar{p}_i'}{\bar{\rho}_i'}\,\delta\rho_i \\ \delta Q_i = \dfrac{\partial g(\rho_i)}{\partial \rho_i}\,\delta\rho_i = \dfrac{\partial \bar{Q}_i}{\partial \bar{\rho}_i}\,\delta\rho_i = \dfrac{\dot{\bar{Q}}_i}{\bar{\rho}_i'}\,\delta\rho_i \end{cases}$$

The first condition is satisfied for all fluids with a definite equation of state. The second one is not true in general. However, as long as perturbations are adiabatic, all space-dependent quantities can be expressed in terms of a unique function of space:

$$\forall i, \begin{cases} \rho_i(x) = \text{function}\,(a(x)) \\ p_i(x) = \text{function}\,(a(x)) \end{cases}$$

So necessarily $Q_i(x) = \text{function}\,(a(x)) = \text{function}\,(\rho_i(x))$
$(= \text{function}\,(\rho_j(x)))$

So there is no choice: the non-adiabatic coupling remains zero if initially one had $\forall i,j \;\; \xi_i = \xi_j$. Hence, even complicated interactions cannot break the relations $S_{ij} = 0$ as long as $k \ll aH$.

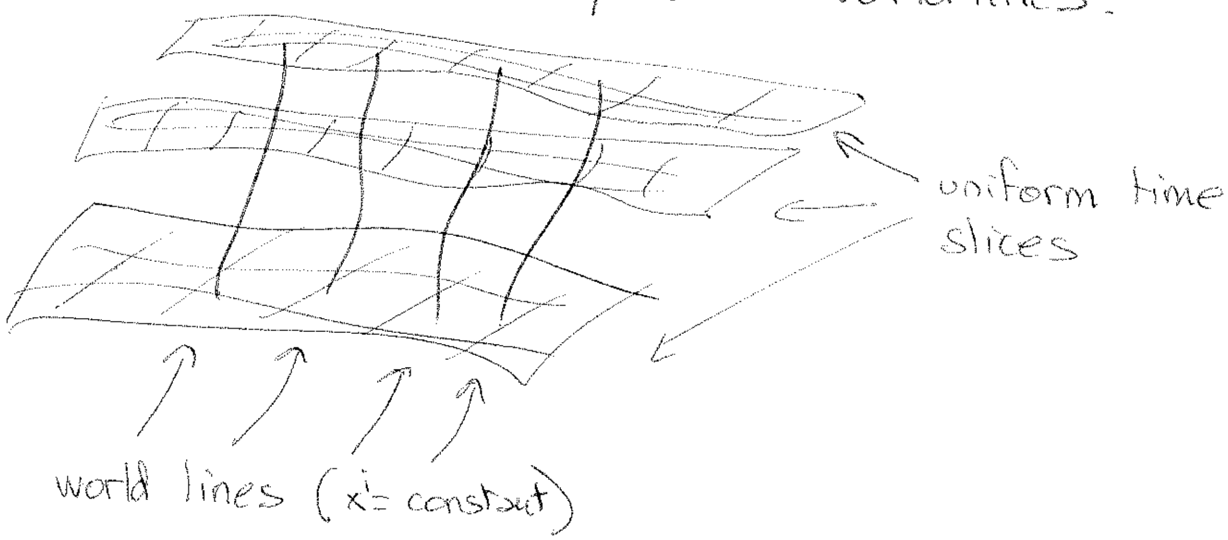
<u>Remark:</u> $S_{ij} = 0$ is a gauge-independent statement:

$$S_{ij} \xrightarrow[\text{transformation}]{\text{gauge}} S_{ij}' + \frac{\varepsilon^0 \dot{\bar{\rho}}_i}{\bar{\rho}_i + \bar{p}_i} - \frac{\varepsilon^0 \dot{\bar{\rho}}_j}{\bar{\rho}_j + \bar{p}_j} = S_{ij}' - 3H\varepsilon^0 \left(\frac{\bar{\rho}_i + \bar{p}_i}{\bar{\rho}_i + \bar{p}_i} - \frac{\bar{\rho}_j + \bar{p}_j}{\bar{\rho}_j + \bar{p}_j} \right) = 0$$

So $S_{ij} = \xi_i - \xi_j$ is gauge-independent (although $\psi - \frac{1}{3}\frac{\delta\rho_i}{\bar{\rho}_i + \bar{p}_i}$ is not).

In summary, all these arguments show that if in the early universe we perturb only one degree of freedom (e.g. the inflaton), then perturbations must be adiabatic; this adiabaticity is then preserved outside the Hubble radius because of causality: it is not possible that a second function of space comes into play on scales $k \ll aH$.

This is very well summarized by the so-called "parallel universe interpretation". In order to understand the universe on super-Hubble scales, one can think of it as a set of independent world lines:



such that along each worldline, the evolution is the same as in homogeneous cosmology. Simply, on each worldline, there is an initial time shift due to the generation of primordial fluctuations: so, on each wordline, all quantities read like

$$\rho_i(\vec{x},t) = \bar{\rho}_i\left(t + \delta t(x)\right)$$

This interpretation is remarkably simple and useful. However:

- ⤳ it does not make sense if the mechanism for generating perturbations leads to two independent functions of space (e.g. if there are 2 inflatons, perturbations cannot be summarized by an initial time shift $\delta t(x)$)
- ⤳ it does not make sense for perturbations inside the horizon for which gradients, shear, etc... are important (i.e. in this case the wordline "talk to each other"; note that also, in that case, $\left\{\frac{T_{\mu\nu}(x)}{g_{\mu\nu}(x)}\right\}$ is non-diagonal in general, while it is in homogeneous cosmology).

IV.3.B) Isocurvature initial conditions

Last sections: we proved that for entropy perturbations

$\left\{ \forall i,j \quad S_{ij}=0 \text{ at time of Hubble exit for a given mode } k \right\}$

$$\Downarrow$$

$\left\{ \forall i,j \quad S_{ij}=0 \text{ for this mode, until Hubble re-entry} \right\}$

So, in order to have at least one $S_{ij} \neq 0$ in the initial conditions (i.e. at $a \sim a_{ini}$ for $k \leq aH$), one must have:

(*) a mechanism for generating $S_{ij} \neq 0$ before Hubble exit in the early universe; this mechanism MUST involve at least two functions of space, so that $\frac{\delta n_i}{n_i} \neq \frac{\delta n_j}{n_j}$

E.g. two light scalar fields during inflation

↳ means: with $m^2 \equiv \frac{\partial^2 V}{\partial \varphi^2} \ll H^2$

Then, this field is associated to a variable ξ_k with $\xi_k'' + (k^2 + \frac{z''}{z})\xi_k = 0$, leading to scale-invariant perturbations $\xi_k \sim \frac{aH}{\sqrt{2k^3}}$, $\delta\varphi \sim \frac{H}{\sqrt{2k^3}}$ (like inflaton in chapter 3). If $m^2 \gg H^2$, then $\xi_k'' + (k^2 + a^2 m^2 \xi_k) = 0 \rightarrow$ perturbations are strongly suppressed

If these two fields have a comparable energy $\rho_1 \sim \rho_2 \sim \rho_{tot}$, they are <u>two inflatons</u>. If $\rho_2 \ll \rho_1$ and $\rho_{tot} \simeq \rho_1$, then φ_1 is the inflaton, and φ_2 the "curvaton" (which fluctuations can play an important role after inflation).
A famous example of such a curvaton is the Peccei-Quinn Axion.

(*) Once $S_{ij} = \xi_i - \xi_j \neq 0$, nothing guarantees that $\dot{\xi}_i = \dot{\xi}_j = 0$ and that S_{ij} survives! In particular, thermal equilibrium enforces $S_{ij} = 0$ for all pairs of species in thermal equilibrium with no chemical potentials. Usually, entropy perturbations S_{ij} survive if one species remains always decoupled or carries a chemical potential.

Typical situation leading to entropy perturbations:

* inflaton φ_1 decays in SM particles (γ, b, leptons)
* $\left\{\begin{matrix} 2^{nd} \text{ inflaton} \\ \text{curvaton} \end{matrix}\right\}$ φ_2 $\left\{\begin{matrix} \text{decays into} \\ \text{consist in} \end{matrix}\right\}$ decoupled CDM particles (never in equilibrium!)

Then, $\frac{\delta n_{cdm}}{n_{cdm}} \neq \frac{\delta n_\gamma}{n_\gamma} = \frac{\delta n_b}{n_b} \Rightarrow \begin{cases} S_{cdm,\gamma} \neq 0 \\ S_{cdm,b} \neq 0 \\ S_{\gamma,b} = 0 \end{cases}$

If there are non-zero entropy perturbations at initial time $a_{ini} \sim 100\, a_{eq}$, we need to incorporate them in the solution for the evolution of cosmological perturbations in the range $a_{ini} \leq a < a_0$

In general, we saw that in presence of N fluids, the coupled system of evolution equation has 2N solutions $\alpha = 1, 2, \dots, 2N$:

$$\forall i = 1, .., N \qquad \delta_i(\vec{k}, z) = \sum_{\alpha=1}^{2N} C_\alpha(\vec{k})\, f_i^\alpha(k, z)$$

where $C_\alpha(\vec{k})$ is the coefficient of the solution α for the wavevector $\vec{k}$. The basis of solutions $f_i^\alpha(k,z)$ of the linear system can ALWAYS be chosen in the following way (proof not given here):

- $\alpha = 1$ is the growing adiabatic mode: $\forall i,j$, $\underline{S_{ij} = 0}$ for $k \ll aH$, and $\underline{\dot{\mathcal{R}} = 0}$ for $k \ll aH$ (it is called "growing" instead of "constant" because in the historical "synchronous gauge", the δ_i's grow with time for $k \ll aH$, instead of remaining constant as in the Newtonian gauge).
- $\alpha = 2, \dots, N$ are such that $S_{ij} \neq 0$, but $\mathcal{R} \xrightarrow[k \to 0]{} 0$ (i.e., curvature perturbations vanish asmptotically outside the Hubble radius): hence, these (N-1) modes are called "isocurvature modes" or "growing isocurvature modes".
- $\alpha = N+1, \dots, 2N$ are all decaying modes.

In minimal cosmological models with no entropy perturbations, only $\alpha=1$ matters. In models with entropy perturbations related to n independent functions of space in the early universe (e.g. n inflatons), there will be $(n-1)$ isocurvature modes turned on in addition to the adiabatic mode. In some particular models, it is even possible to have a null or negligible adiabatic mode, and significant isocurvature ones.

The initial conditions are then specified by the power spectrum of each non-zero $C_\alpha(\vec{k})$, plus a possible correlation angle between them.

<u>Example:</u> Inflaton 1 $\xrightarrow{\text{decays in}}$ γ, b, leptons

Inflaton 2 $\longrightarrow$ cdm

Then, at a_{ini}, when $k \ll aH$:

"function of $\delta\varphi_1$ during inflat

$$\xi_\gamma = \xi_b = f(\delta\varphi_1)$$
$$\xi_{cdm} = g(\delta\varphi_2)$$

function of "$\delta\varphi_2$ during inflation

So, the adiabatic mode ($\xi_\gamma = \xi_b = \xi_{cdm}$) is seeded by a linear combination of $\delta\varphi_1^{infl}$ and $\delta\varphi_2^{infl}$. Another linear combination seeds one isocurvature mode. Hence, $C_1(\vec{k})$ and $C_2(\vec{k})$ (coefficients of adiabatic and isocurvature modes) are not statistically independent, Although $\delta\varphi_1$ and $\delta\varphi_2$ are independent. Then, the initial condition consists in three functions:

$$\mathcal{S}_{C_1}(k)\,,\quad \mathcal{S}_{C_2}(k)\,,\quad \Theta(k)$$

↳ correlation angle

which can be computed within a given inflationary + early universe model. The perturbations, e.g., of CD matter today reads:

$$\delta_{cdm}(\vec{k},t_0) = C_{ad}(\vec{k})\, f^{ad}_{cdm}(k,t_0) + C_{iso}(\vec{k})\, f^{iso}_{cdm}(k,t_0)$$

so the power spectrum reads:

$$\begin{aligned}\mathcal{S}_{cdm}(k,t_0) = {} & \mathcal{S}_{C_{ad}}(k,t_0)\,|f^{ad}_{cdm}(k,t_0)|^2 \\ & + \mathcal{S}_{C_{iso}}(k,t_0)\,|f^{iso}_{cdm}(k,t_0)|^2 \\ & + 2\left[\mathcal{S}_{C_{ad}}(k,t_0)\,\mathcal{S}_{C_{iso}}(k,t_0)\right]^{1/2} \underbrace{\cos\Theta(k)}_{\text{correlation angle}} \mathrm{Re}\left[f^{ad}_{cdm}(k,t_0)\, f^{iso\,*}_{cdm}(k,t_0)\right]\end{aligned}$$

HOWEVER, a <u>generic</u> result is that $\{f^{iso}_i(k,z)\}$ solutions lead to observables which contradict observations (acoustic oscillations with the wrong phase $\Rightarrow$ wrong position of CMB peaks). Hence, current observations prove that isocurvature modes are either null or negligible in our Universe.

As a consequence, <u>we will not consider entropy/isocurvature perturbations anymore in this course</u>.

IV.3.C. Time evolution of growing adiabatic mode on super-Hubble scales:

We come back to the Einstein equations (eqs. (4), (5), (6), (7) of section II.4.) and write them in Fourier space:

(I) $-3\left(\frac{a'}{a}\right)^2\phi - 3\frac{a'}{a}\psi' - k^2\psi = 4\pi G a^2 \delta\rho$ ← $\delta\rho$ total

(II) $-k^2\left(\frac{a'}{a}\phi + \psi'\right) = 4\pi G a^2 (\bar\rho + \bar p)\,\theta$ ← total $\bar\rho, \bar p$ and θ

(III) $\left(2\left(\frac{a''}{a}\right) - \left(\frac{a'}{a}\right)^2\right)\phi + \frac{a'}{a}(\phi' + 2\psi') + \psi'' - \frac{k^2}{3}(\phi - \psi) = 4\pi G a^2 \delta p$ ← total pressure perturba[tion]

(IV) $k^2(\phi - \psi) = 12\pi G a^2 (\bar\rho + \bar p)\,\sigma$ ← total shea[r]

For perfect fluids, the energy-momentum tensor has $\sigma = 0$ (perfect fluids have a bulk velocity and no anisotropic pressure). This is the case for photons + baryons when they are tightly coupled to each other, and also for CDM (non-relativistic collisionless $\Leftrightarrow$ pressureless fluid)

Hence, σ can receive contributions from:

- γ's after τ_{dec}: but we don't care, after τ_{dec} $\rho_\gamma \ll \rho_{tot}$ and photons play no role in Einstein equations
- ν's at all times $\tau > \tau_{ini}$: their shear is significant, but <u>in this chapter, we will neglect neutrinos for simplicity: this allows us to write $\sigma = 0$, leading to important simplifications</u>

Then (IV) $\Rightarrow k^2(\phi-\psi)=0$ (or $\Delta(\phi-\psi)=0$):
either $\phi=\psi$, or $\phi-\psi=$ linear function of x^i diverging at infinity... hence $\boxed{\phi=\psi \text{ in this approximation.}}$

We eliminate ψ from Einstein equation and search for an equation of evolution for ϕ, knowing that $\mathring{\mathcal{R}}=0$ for the growing adiabatic mode. We write (I) in the $k \ll aH$ limit:

$$-3\left(\frac{a'}{a}\right)^2\phi - 3\frac{a'}{a}\phi' = 4\pi G a^2 \delta\rho$$

We know that:

$$\mathcal{R} = \phi - \frac{1}{3}\frac{\delta\rho}{\rho+p}$$

Using Friedman ($H^2 = \frac{a'^2}{a^4} = \frac{8\pi G}{3}\rho$) and the background energy conservation equation ($\rho' = -3\frac{a'}{a}(\rho+p)$), we can combine the two above relations into a 1st order linear, inhomogeneous equation for ϕ:

$$3\left(\frac{a'}{a}\right)^2\phi - 3\frac{a'}{a}\phi' = -4\pi G a^2 \rho' \frac{a}{a'}(\mathcal{R}-\phi)$$

$$\Leftrightarrow \left[-\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2\right]\phi + \frac{a'}{a}\phi' = \left(-\frac{a''}{a} + 2\left(\frac{a'}{a}\right)^2\right)\mathcal{R}$$

General solution of homogeneous equation: $\phi \propto \frac{a'}{a^3} = \frac{H}{a}$ (decaying mode)

Full solution when $\mathcal{R}=$cte:

$$\boxed{\phi = \mathcal{R}\left(1 - \frac{a'}{a^3}\int_{\tau_1} a^2 d\tau\right) = \mathcal{R}\left(1 - \frac{H}{a}\int_{t_1} a\,dt\right)}$$ exact solution for $k=0$

where τ_1 (or t_1) is the constant of integration.

We can now compute the relation between Φ and $\mathcal{R}$ during the various stages:

$$\begin{cases} \text{RD}: a \propto t^{1/2} \Rightarrow \Phi = \mathcal{R}\left(1 - \frac{1}{2} t^{-3/2} \int_{t_1}^{t} t^{1/2} dt\right) = \frac{2}{3}\mathcal{R} + \text{decaying mode } \propto t^{-3/2} \\ \text{MD}: a \propto t^{2/3} \Rightarrow \Phi = \mathcal{R}\left(1 - \frac{2}{3} t^{-5/3} \int_{t_1}^{t} t^{2/3} dt\right) = \frac{3}{5}\mathcal{R} + \text{decaying mode } \propto t^{-5/3} \\ \Lambda\text{D}: \text{no trivial solution; } \Phi \text{ decays} \end{cases}$$

Hence Φ is constant during RD, decreases by $\frac{9}{10}$ at equality, remains constant during MD, and decays during ΛD. Let us know infer the evolution of δ_i's.

$$(\mathrm{I}) \Rightarrow -3\left(\frac{a'}{a}\right)^2 \Phi - 3\frac{a'}{a}\Phi' = 4\pi G a^2 \rho_{tot} \frac{\delta\rho_{tot}}{\rho_{tot}} \quad \text{for } k \ll aH$$

Using $\Phi = cte$ and Friedmann gives:

$$-3\left(\frac{a'}{a}\right)^2 \Phi = \frac{3}{2}\left(\frac{a'}{a}\right)^2 \delta_{tot} \Leftrightarrow \delta_{tot} = -2\Phi \quad \text{for } k \ll aH$$

minus sign: $\Phi < 0 \Leftrightarrow$ potential well $\Leftrightarrow$ overdensity $\delta_{tot} > 0$

RD: $\underbrace{\rho_r}_{\text{radiation}} \equiv \rho_\gamma \gg \underbrace{\rho_m}_{\text{matter}} \equiv \underbrace{\rho_b}_{\text{baryons}} + \underbrace{\rho_c}_{\text{cdm}}$

So $\delta_{tot} = \dfrac{\rho_r \delta_r + \rho_m \delta_m}{\rho_r + \rho_m} \simeq \delta_r$; so $\delta_r = -2\Phi$

Growing adiabatic mode during RD:

$$\boxed{-2\Phi = \delta_r = \frac{4}{3}\delta_m = -\frac{4}{3}\mathcal{R} \quad \text{for } k \ll aH}$$

(with $\delta_r \equiv \delta_\gamma$, $\delta_m \equiv \delta_b = \delta_c$)

MD $\rho_r \ll \rho_m$, so $\delta_{tot} \simeq \delta_m$; so $\delta_m = -2\phi$

Growing adiabatic mode during MD:

$$\boxed{-2\phi = \delta_m = \frac{3}{4}\delta_r = -\frac{6}{5}\mathcal{R} \quad \text{for } k \ll aH}$$

ΛD $\rho_r \ll \rho_m$, so $\delta_{tot} \simeq \delta_m$ in absence of perturbations for Λ (or some dark energy).

Growing adiabatic mode during ΛD:

$$\boxed{-2\phi = \delta_m = \frac{3}{4}\delta_r \text{ decays with time}}$$

Note that during RD or MD, we are now able to relate the power spectrum of ϕ or of each δ_i for $k \ll aH$ to the power spectrum of $\mathcal{R}$ computed in Chapter III for single-field inflation. E.g. during RD:

$$\mathcal{S}_\phi(k) \equiv \frac{k^{-3}}{2\pi^2}\langle|\phi(\vec{k})|^2\rangle = \frac{k^{-3}}{2\pi^2}\left(\frac{2}{3}\right)^2\langle|\mathcal{R}(\vec{k})|^2\rangle = \frac{4}{9}\mathcal{S}_{\mathcal{R}}(k)$$

Hence, $\mathcal{S}_{\mathcal{R}}(k)$ computed during inflation gives the initial conditions for ϕ, δ_i's, ... during radiation domination when $k \ll aH$!