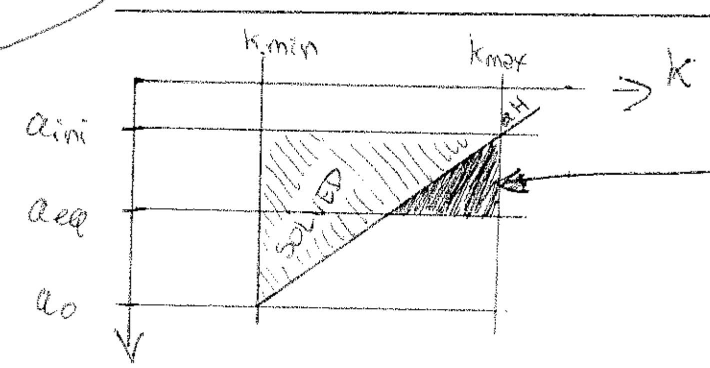
IV.4. Sub-Hubble evolution during RD



we are interested in this triangle

Until know we avoided to write the equation of motion of each fluid (i) coming from $D_\mu T^{(i)\mu}_{\ \ \nu} = 0$ or coupling term, at order 1 in perturbations. Let us do it only for a decoupled perfect fluid. Then $\delta T^{(i)\mu}_{\ \ \nu}$ contains three degrees of freedom coupled with scalar metric perturbations $\delta\rho_i$, δp_i an θ_i. It can be shown that $\sigma_i = 0$ for a perfect fluid (bulk velocity $\Rightarrow$ no anisotropic pressure).

A single fluid has an equation of state $p_i = p_i(\rho_i)$.

So $\delta p_i = c_s^2 \delta\rho_i$ with $c_s^2 \equiv \frac{\partial p_i}{\partial \rho_i} = \frac{\dot{\bar p}_i}{\dot{\bar\rho}_i}$ (sound speed squared)

If $p_i = w_i \rho_i$, then $c_s^2 = w_i$.

For such a fluid (assuming $\dot w_i = 0$):

$$D_\mu T^{(i)\mu}_{\ \ 0} \Rightarrow \delta_i' = (1+w_i)(\theta_i + 3\psi') \qquad \text{(Continuity equation)}$$

$$D_\mu T^{(i)\mu}_{\ \ i} \Rightarrow \theta_i' = \frac{a'}{a}(3w_i - 1)\theta_i - k^2\phi - \frac{w_i}{1+w_i} k^2 \delta_i \qquad \text{(Euler Equation)}$$

<u>Remark</u>: if $k \to 0$, Euler $\Rightarrow \{\theta_i \to 0\}$ and Continuity implies

$$\delta_i' = (1+w_i)3\psi' \Rightarrow \left(\psi - \frac{1}{3}\frac{\delta_i}{1+w_i}\right)' = 0 \Rightarrow \zeta_i' = 0$$ in agreement with previous sections

Application to the problem at hand:

* CDM $\Leftrightarrow$ pressureless fluid $(w=0)$, so

$$\boxed{\begin{aligned}\delta_c' &= \theta_c + 3\psi' \\ \theta_c' &= -\frac{a'}{a}\theta_c - k^2\phi\end{aligned}}$$

* during RD, baryons and photons are tightly coupled. Their perturbations are related:

Thermal equilibrium $\Rightarrow \rho_\gamma \propto T^4,\ \rho_b \propto T^3$, so $\delta_\gamma = \frac{4}{3}\delta_b = 4\frac{\delta T}{T}$

The sound speed in this fluid follows from:

$$c_s^2 = \frac{\partial P_{\{\gamma+b\}}}{\partial \rho_{\{\gamma+b\}}} = \frac{\dot{P}_\gamma + \dot{P}_b}{\dot{\rho}_\gamma + \dot{\rho}_b} = \frac{\frac{1}{3}\dot{\rho}_\gamma + 0}{\dot{\rho}_\gamma + \dot{\rho}_b} = \frac{1}{3}\,\frac{\frac{4}{3}\rho_\gamma}{\frac{4}{3}\rho_\gamma + \rho_b}$$

$$\text{So } c_s^2 = \frac{1}{3}\,\frac{1}{1+\frac{3}{4}\frac{\rho_b}{\rho_\gamma}} \simeq \frac{1}{3} \text{ during RD } (\rho_b \ll \rho_\gamma)$$

The density of this fluid follows from:

$$\delta_r = \frac{\rho_\gamma\delta_\gamma + \rho_b\delta_b}{\rho_\gamma + \rho_b} \simeq \delta_\gamma \text{ during RD } (\rho_b \ll \rho_\gamma)$$

Hence we treat $\gamma + b$ as a single fluid "r" with sound speed $c_s^2 = \frac{1}{3} = w$ (individual perturbations then follow from $\delta_\gamma = \delta_r,\ \delta_b = \frac{3}{4}\delta_r$). So:

$$\boxed{\begin{aligned}\delta_r' &= \frac{4}{3}(\theta_r + 3\psi') \\ \theta_r' &= -k^2\phi - \frac{k^2}{4}\delta_r\end{aligned}}$$

* again we neglect neutrinos $\Rightarrow \Phi = \Psi$

Hence we have 5 variables $(\delta_r, \theta_r, \delta_c, \theta_c, \Phi)$ and enough equations of motion...

Let us give the solutions in the limit $\rho_c \ll \rho_r$ valid during RD. In this limit,

$$\delta_{tot} = \frac{\rho_r \delta_r + \rho_c \delta_c}{\rho_r + \rho_c} \simeq \delta_r$$

and <u>CDM plays the role of a test fluid, while "r" is self-gravitating.</u>

Then, Einstein (I) gives:

$$-3\left(\frac{a'}{a}\right)^2 \Phi - 3\frac{a'}{a}\Phi' - k^2\Phi = 4\pi G a^2 \delta\rho_{tot}$$

Einstein (III) gives:

$$\left(2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right)\Phi + 3\frac{a'}{a}\Phi' + \Phi'' = 4\pi G a^2 \delta p_{tot}$$

Using $\delta p_{tot} = c_s^2 \delta\rho_{tot}$, we compute $c_s^2 \times (\mathrm{I}) - (\mathrm{III})$:

$$\boxed{\left(2\frac{a''}{a} - (1-3c_s^2)\left(\frac{a'}{a}\right)^2\right)\Phi + k^2 c_s^2 \Phi + 3(1+c_s^2)\frac{a'}{a}\Phi' + \Phi'' = 0}$$

During RD, $\delta\rho_{tot} \simeq \delta\rho_r$, $\delta p_{tot} \simeq \delta p_r$ and $c_s^2 \simeq \frac{1}{3}$. So:

$$2\frac{a''}{a}\Phi + \frac{k^2}{3}\Phi + 4\frac{a'}{a}\Phi' + \Phi'' = 0$$

Moreover $a \propto t^{1/2} \Rightarrow a \propto \tau$: so $\frac{a'}{a} = \frac{1}{\tau}$ and $\frac{a''}{a} = 0$:

$$\frac{k^2}{3}\Phi + \frac{4}{\tau}\Phi' + \Phi'' = 0$$

Solutions: $\Phi \equiv \tau^{3/2} v \Rightarrow v = J_{\pm\frac{3}{2}}(\underbrace{k c_s \tau}_{\frac{k}{\sqrt{3}}\tau})$ (Bessel functions)

Finally: $\Phi = C_1(\vec{k})\,(k\tau)^{-2}\left(\frac{\sin kc_s\tau}{kc_s\tau} - \cos kc_s\tau\right)$
$+ C_2(\vec{k})\,(k\tau)^{-2}\left(-\frac{\cos kc_s\tau}{kc_s\tau} - \sin kc_s\tau\right)$

for each $\vec{k}$ during RD.

Interpretation :

(*) This solution is of the general form $\delta_i(\vec{k},t) = \sum_\alpha C_\alpha(\vec{k})\, f_i^\alpha(k,\tau)$ used in previous sections. Only two solutions here, because we considered a single self-gravitating fluid "r" coupled with metric perturbations.

(*) $C_1(\vec{k})$ and $C_2(\vec{k})$ can be viewed as 2 stochastic numbers.

(*) solutions are oscillatory ; in the b+γ fluid, the competition between gravity and photon pressure leads to the propagation of acoustic oscillations (whenever the system is placed initially out of equilibrium : this is the case since for $k \ll aH$, $\Phi = -\frac{1}{2}\delta_\gamma \neq 0$)

(*) argument of cos( ), sin( ) $\equiv kc_s\tau$. Related to sound horizon: $d_s \equiv a\int_{t_i}^{t}\frac{dt\, c_s}{a} = a\int_{\tau_i}^{\tau} c_s d\tau = a\, c_s(\tau-\tau_i)$
$\simeq a c_s \tau$ if $\tau \gg \tau_i$

mode inside $d_s \Leftrightarrow \lambda \leq d_s \Leftrightarrow a\frac{2\pi}{k} \leq a c_s\tau \Leftrightarrow \frac{kc_s\tau}{2\pi} \geq 1$

So $\cos(kc_s\tau) = \cos\left(\frac{kc_s\tau}{2\pi}2\pi\right) = \cos\left(2\pi\frac{d_s}{\lambda}\right)$

Mode starts to oscillate only inside d_s for causality reasons! $\frac{d_s}{\lambda} \sim$ # of period of oscillations

* limit $k\tau \to 0$: expansion of previous solution at order $\mathcal{O}(k\tau)^3$ gives:

$$\Phi(\vec{k}, \tau \to 0) = \frac{1}{9} C_1(\vec{k}) - \frac{\sqrt{3}}{(k\tau)^3} C_2(\vec{k})$$

Hence we identify C_1 to the growing adiabatic mode of previous sections, and C_2 to a decaying mode. The initial conditions near τ_{ini} can be taken as:

  * $C_2 = 0$ (since this mode has decayed considerably since inflation!!!)
  * $C_1(\vec{k})$ = gaussian stochastic number with power spectrum
    $\mathcal{P}_{C_1}(k) = 36\, \mathcal{P}_{\mathcal{R}}(k)$
    such that:
    $\mathcal{P}_\Phi(k) = \frac{1}{81} \mathcal{P}_{C_1}(k) = \frac{4}{9} \mathcal{P}_{\mathcal{R}}(k)$
    as derived in section IV.3.C.

* we can infer δ perturbations from Einstein:

(I) $\Rightarrow -\frac{3}{\tau^2}\Phi - \frac{3}{\tau}\Phi' - k^2\Phi = \frac{3}{2\tau^2}\delta_r$

Solution written in two pages; in summary, δ_r oscillates, and $\delta_r \xrightarrow[k\tau \to 0]{} -\frac{2}{9} C_1 = -2\Phi$ as expected

* opposite limit:

$$\Phi \xrightarrow[k\tau \to \infty]{} -C_1(\vec{k})(k\tau)^{-2} \cos k c_s \tau$$

$$\delta_r \xrightarrow[k\tau \to \infty]{} \frac{2}{3} C_1(\vec{k}) \cos k c_s \tau$$

Hence, inside sound horizon, δ_r experiences acoustic oscillations of constant amplitude. Oscillations of ϕ are damped, but this is consistent with Poisson equation ($\Leftrightarrow$ Einstein (I) in limit $k \gg aH$):

$$\underbrace{\frac{\Delta}{a^2}}_{\substack{\text{physical}\\ \text{Laplacian}}} \phi = 4\pi G \delta\rho_{tot} \Leftrightarrow -k^2 \phi = 4\pi G a^2 \delta\rho_{tot}$$

Here: $-k^2\phi = \underbrace{4\pi G a^2 \rho_r}_{\substack{\text{decays like}\\ a^{+2-4} = a^{-2}}} \delta_r$

So constant oscillations of $\delta_r \Rightarrow$ damped oscillations of ϕ

(*) δ_c can be computed from $\begin{cases} \delta_c' = \theta_c + 3\phi' \\ \theta_c' = -\frac{1}{\tau}\theta_c - k^2\phi \end{cases}$

Can be combined in 2nd order inhomogeneous eq. for δ_c.
Two solutions: growing and decaying mode. Growing mode reads: $\boxed{\delta_c = [-C_\phi(\vec{k}) \ln(k c_s \tau) + cte]}$ valid only for $k \ll aH$

Hence matter perturbations grow slowly (more slowly than when CDM will be self-gravitating!)

NEXT PAGE SUMMARISES ALL THE RD EVOLUTION.

Perturbation evolution during radiation domination (simplest approximation)

Limit of:

- perfect tight-coupling between baryons and photons: $\delta_\gamma = 4\frac{\delta T}{T} = \frac{4}{3}\delta_b \equiv \delta_r$, where "r" is a perfect fluid ($\sigma_r = 0$) with sound speed c_s;
- negligible matter density: $\rho_b \ll \rho_\gamma$ (implies $c_s^2 = 1/3$ for the fluid "r"), and $\rho_c \ll \rho_\gamma$ (CDM is then a test fluid, while "r" is self-gravitating);
- no neutrinos (implies $\sigma_{\rm tot} = 0$ and $\phi = \psi$)

In this limit in Newtonian gauge:

$$\phi = C_1(k\tau)^{-2}\left(\frac{\sin z}{z} - \cos z\right) + C_2(k\tau)^{-2}\left(-\frac{\cos z}{z} - \sin z\right) \qquad (1)$$

with $z \equiv c_s k\tau = 2\pi\frac{d_s(\tau)}{\lambda(\tau)}$;

$$\begin{aligned}\delta_\gamma \;=\; & -2C_1(k\tau)^{-2}\left[2(z^2-1)\frac{\sin z}{z} - (z^2-2)\cos z\right] \\ & +2C_2(k\tau)^{-2}\left[2(z^2-1)\frac{\cos z}{z} - (z^2-2)\sin z\right] . \end{aligned} \qquad (2)$$

In the limit $k\tau \ll 1$ one has:

$$\phi \longrightarrow \frac{1}{9}C_1 - \frac{\sqrt{3}}{(k\tau)^3}C_2 \qquad (3)$$

so C_1 is the coefficient of the growing adiabatic mode, for which $\delta_\gamma = \delta_r = \frac{4}{3}\delta_b = \frac{4}{3}\delta_c = -2\phi = -\frac{4}{3}\mathcal{R} = -\frac{2}{9}C_1$ (so, it is called "growing", but in the newtonian gauge and on super-Hubble scales it is actually constant).

For CDM, no simple analytic expression, but simple asymptotic behaviour: for $k\tau \ll 1$ see above ($\delta_c = -\frac{1}{6}C_1$); for $k\tau \gg 1$, $\delta_c \longrightarrow -C_1\log(k\tau)$.

The figure shows the evolution of all modes k at a given time τ, assuming a completely arbitrary initial condition: $C_1(\vec{k}) = -9$ for all modes. The same figure can also be interpreted as the time evolution of a single mode k starting from this initial condition.