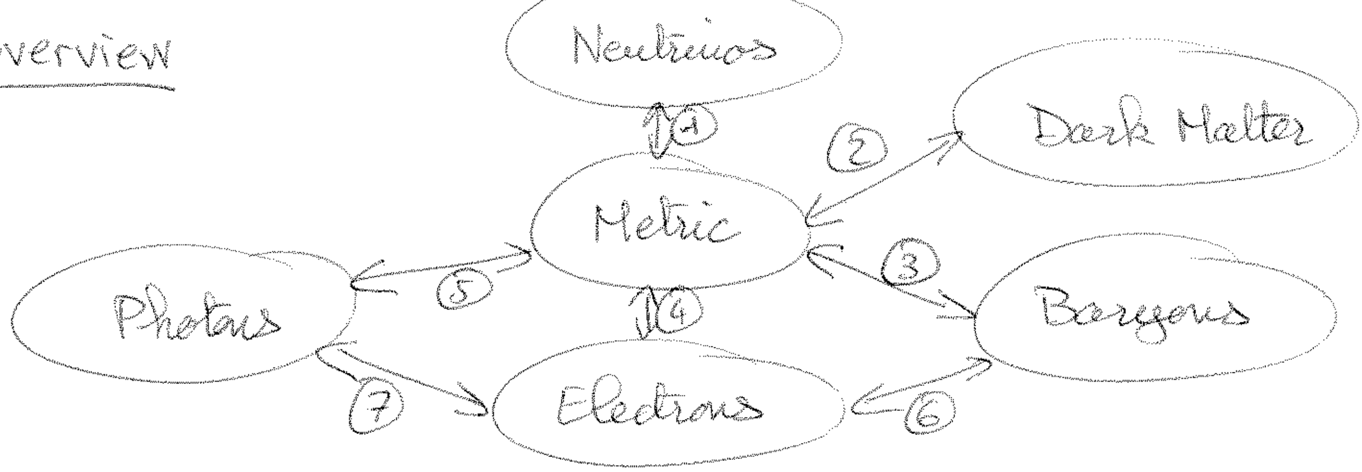
Chapter V CMB Anisotropies

Overview



Each "⟺" stands for a coupling.
"Photon ⟺ baryon" exists, but negligible wrt "photon ⟺ electr

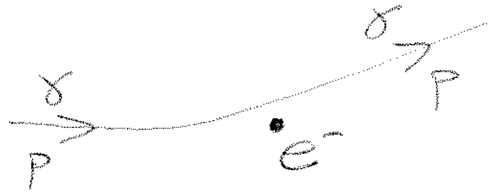
1 : we will ignore this gravitational coupling because for simplicity, we neglect neutrinos in this Chapter

2, 3 : correcty treated in Chapter IV

5 : in Chapter IV we treated photons as a fluid; we must derive a more correct formalism based on a phase-space distribution function, and find an equation of evolution including the coupling 5 with the metric and 7 with the electrons.

4 : really negligible. Electrons very light w.r.t. baryons ⇒ no significant back-reaction on metric. Electron only play the role of "mediator" btw photons and baryons.

⑦ Scattering between γ and e^-. In principle, this is Compton scattering. Here, e^- are non-relativistic, photon momentum nearly conserved, scattering nearly elastic: limit of <u>Thomson scattering</u>.



⑧ Coulomb scattering between e^- and nuclei (for simplicity, protons). Very efficient, ensures charge conservation locally (ie, in the limit in which all b would be protons, $\frac{\delta n_{e^-}}{n_{e^-}} = \frac{\delta n_p}{n_p}$) and common bulk velocities: $\theta_{e^-} = \theta_b$.

At background level, photons described by Bose-Einstein distribution $f^0(\eta, p) = \frac{1}{e^{p/T(\eta)} + 1}$

⚠ CONFORMAL TIME NOTED η IN THIS CHAPTER

At level of perturbation, we expect $f = f^0 + f^1$ with $f^1(\eta, x^i, p, \hat{n}^i)$.

Two possibilities:

* spectral distorsion: f does not depend on p as a blackbody
* blackbody spectrum with perturbations and anisotropies in the temperature:

$$T(\eta) \longrightarrow T(\eta) + \delta T(\eta, x^i, \underset{\substack{\uparrow \\ \text{unit vector} \\ \text{(direction)}}}{\hat{n}^i})$$

We will see that only the 2nd case happens, because: until decoupling, photon $\Leftrightarrow$ electron in thermal equilibrium

↳ f = blackbody in each point, in electron rest-frame : $\delta T = \delta T(\eta, x^i)$

↳ additional dipolar dependance of δT in other frames : $\delta T = \delta T(\eta, x^i, \hat{n}^i)$

During and after decoupling, the Thomson and gravitational interactions <u>preserve</u> this shape.

So the problem reduces to finding equations of evolution for $\frac{\delta T}{T}(\eta, x^i, \hat{n}^i)$
(instead of $f(\eta, x^i, p, \hat{n}^i)$ in general ; instead of δ_γ and $\theta_\gamma(\eta, x^i)$ in over-simplification of Chapter IV).

V.1. Boltzmann equation for photons

In general, $f = f(x^\mu, \underline{P}^\mu)$

phase-space distribution

position

conjugate 4-momentum

$\underline{P}^\mu = \frac{dx^\mu}{d\lambda}$ with $\underline{P}_\mu \underline{P}^\mu (= g_{\mu\nu} \underline{P}^\mu \underline{P}^\nu) = 0$

for massless particles like photons.

We define <u>momentum</u> as $p^2 \equiv g_{00} P^0 P^0 = -g_{ij} P^i P^j$

In Newtonian gauge:

$$p^2 = a^2 (1 - 2\psi) \delta_{ij} \underline{P}^i \underline{P}^j = a^2 (1 + 2\phi) \underline{P}^{0\,2}$$

So we can eliminate dependance on $\underline{P}^0$:

$f = f(\eta, x^i, \underline{P}^i)$

conformal time

Interpretation: number of particles with given momentum and position range given by

$$dN = f(\eta, x^i, \underline{P}^i) d^3x^i \, d^3\underline{P}^i$$

We define $\hat{n}_i$ as the photon direction of propagation:

$\underline{P}^i \equiv \sqrt{\delta_{ij} \underline{P}^i \underline{P}^j} \; \hat{n}^i$

↳ normalized so that $\delta_{ij} \hat{n}^i \hat{n}^j = 1$

Then $f = f(\underbrace{\eta, x^i, p, \hat{n}^i}_{1+3+1+2 = 7 \text{ d.o.f.}})$

In general, Boltzmann equation reads:

$$\underbrace{\frac{df}{d\eta}}_{\text{total derivative}} = \underbrace{C[f]}_{\text{collision term, depending on } f \text{ itself (but also on other species distributions)}}$$

So: $$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i}\frac{dx^i}{d\eta} + \frac{\partial f}{\partial p}\frac{dp}{d\eta} + \frac{\partial f}{\partial \hat{n}_i}\frac{d\hat{n}_i}{d\eta} = C[f]$$

We want this equation at <u>order one</u> in perturbations.

Since $f = \underbrace{f^0(\eta, p)}_{\text{background}} + \underbrace{f^1(\eta, x^i, p, \hat{n}^i)}_{1^{st} \text{ order}}$,

$\frac{\partial f}{\partial \eta}$ contains order $0, 1$

$\frac{\partial f}{\partial x^i}$ " " 1

$\frac{\partial f}{\partial p}$ " " $0, 1$

$\frac{\partial f}{\partial \hat{n}^i}$ " " 1

So, we need to evaluate $\frac{dx^i}{d\eta}$ at order 0 only, while $\frac{dp}{d\eta}$ is needed at order 1.

In addition, $\frac{d\hat{n}_i}{d\eta}$ vanishes at order 0 since in <u>unperturbed</u> universe, photons travel in straight line.

Hence $\frac{\partial f}{\partial \hat{n}^i}\frac{d\hat{n}^i}{d\eta}$ is at least of order 2.

It is easy to compute $\frac{dx^i}{d\eta}$:

$$\frac{dx^i}{d\eta} = \frac{dx^i}{d\lambda}\frac{d\lambda}{d\eta} = \frac{p^i}{p^0} \underset{\substack{\text{at} \\ \text{order } 0 \\ !!}}{=} \frac{p^i}{\sqrt{\delta_{ij} p^i p^j}} = \hat{n}^i$$

It is more involved to compute $\frac{dp}{d\eta}$. The geodesics equation gives:

$$\frac{dP^0}{d\lambda} = -\Gamma^0_{\mu\nu} P^\mu P^\nu$$

$' \equiv \frac{d}{d\eta}$

Left-hand side : $\frac{dP^0}{d\lambda} = \frac{dP^0}{d\eta}\frac{d\eta}{d\lambda} = P^{0'} P^0$ can be expressed in terms of $p' = \frac{dp}{d\eta}$

Right-hand side: can be computed in Newtonian gauge.

After a few computational steps, one obtains:

$$\frac{dp}{d\eta} = p\left[-\frac{a'}{a} + \psi' - \hat{n}_i \partial_i \phi\right]$$

Note that:

* at order 0 : $\frac{d \ln p}{d\eta} = -\frac{a'}{a} \Rightarrow p \propto \frac{1}{a}$: redshift of each photon with expansion
* at order 1 : $\frac{d \ln p}{d\eta}$ = function <u>not</u> depending on p. Photons with $\neq p$ crossing same potential well receive the <u>same relative redshifting</u>

Consequence: gravitational interactions <u>cannot distort blackbody spectrum</u>.

Finally Boltzmann equation reads (at order 1):

$$\frac{\partial f}{\partial \eta} + \hat{n}_i \frac{\partial f}{\partial x^i} + p\left[-\frac{a'}{a} + \psi' - \hat{n}_i \partial_i \phi\right]\frac{\partial f}{\partial p} = C[f]$$

At order 0: we replace f by $f^0(\eta, p)$ and get:

$$\frac{\partial f^0}{\partial \eta} - p\frac{a'}{a}\frac{\partial f^0}{\partial \eta} = C[f]$$

But $f^0(\eta,p) = \frac{1}{e^{p/T(\eta)} - 1}$. Replacing, we get:

$$-\left(\frac{T'}{T}+\frac{a'}{a}\right)\frac{p}{T}\,e^{p/T} f^{o\,2} = C[f]$$

After $e^- - e^+$ annihilation, we know that $T \propto \frac{1}{a}$: so the left-hand side vanishes. This shows that at order 0, as soon as $T \propto \frac{1}{a}$, the photon reach an equilibrium corresponding to:

$$C[f]^{(0)} = \text{creation rate} - \text{annihilation rate} = 0$$

So, $C[f]$ will be non-zero only at order 1.

V 1.2. Collisionless equation

After η_{dec} (decoupling time), we can approximate $C[f] = 0$ at any order.

Let us check that full Boltzmann equation has solutions of the form (preserving black body):

$$f(\eta, x^i, p, \hat{n}_i) = \left\{ \exp\left[p \big/ \left(\bar{T}(\eta)\left[1 + \Theta(\eta, x^i, \hat{n}^i)\right]\right)\right] - 1 \right\}$$

$$\left(\text{i.e.,} \quad f = \frac{1}{e^{p/T(1+\Theta)} - 1}\right)$$

where we introduced $\Theta(\eta, x^i, \hat{n}^i) \equiv \frac{\delta T}{\bar{T}}(\eta, x^i, \hat{n}^i)$

= temperature perturbation in point x^i and direction $\hat{n}^i$

We can insert this ansatz in Boltzmann:

$$\underbrace{\frac{(\bar{T}'(1+\Theta)+\bar{T}\Theta')p}{\bar{T}^2(1+\Theta)^2} e^{\frac{p}{\bar{T}(1+\Theta)}} f^2}_{\frac{\partial f}{\partial \eta}} + \hat{n}^i \underbrace{\bar{T}\frac{\partial \Theta}{\partial x^i}\frac{p}{\bar{T}^2(1+\Theta)^2} e^{\frac{p}{\bar{T}(1+\Theta)}} f^2}_{\frac{\partial f}{\partial x^i}}$$

$$-p\left[-\frac{a'}{a}+\psi'-\hat{n}^i\partial_i\phi\right]\underbrace{\frac{1}{\bar{T}(1+\Theta)} e^{\frac{p}{\bar{T}(1+\Theta)}} f^2}_{\frac{\partial f}{\partial p}} = 0$$

After simplification by factor $\frac{p}{\bar{T}(1+\Theta)^2} e^{\frac{p}{\bar{T}(1+\Theta)}} f^2$:

$$-\bar{T}'(1+\Theta)-\bar{T}(\Theta')-\hat{n}^i\bar{T}\frac{\partial\Theta}{\partial x^i}+\left[-\frac{a'}{a}+\psi'-\hat{n}_i\partial_i\phi\right]\bar{T}(1+\Theta)=0$$

* order 0: $-\bar{T}'-\bar{T}\frac{a'}{a}=0 \quad (\bar{T}\propto \frac{1}{a})$

* order 1:

$$\boxed{\Theta'+\hat{n}^i\partial_i\Theta=\psi'-\hat{n}^i\partial_i\phi} \qquad \partial_i=\frac{\partial}{\partial x^i}$$

Equation can be written in terms of $\frac{d\Theta}{d\eta}$:

→ we know that $\frac{d\Theta}{d\eta}=\Theta'+\frac{\partial\Theta}{\partial x^i}\underbrace{\frac{dx^i}{d\eta}}_{\hat{n}^i}+\underbrace{\frac{\partial\Theta}{\partial\hat{n}^i}\frac{d\hat{n}^i}{d\eta}}_{\text{vanishes at order 1}}=\Theta'+\hat{n}^i\frac{\partial\Theta}{\partial x^i}$

→ for the variable $\phi=\phi(\eta,x^i)$ (no dependence on $\hat{n}^i$):

$$\frac{d\phi}{d\eta}=\phi'+\frac{\partial\phi}{\partial x^i}\frac{dx^i}{d\eta}=\phi'+\hat{n}^i\partial_i\phi \text{ along photon geodesics}$$

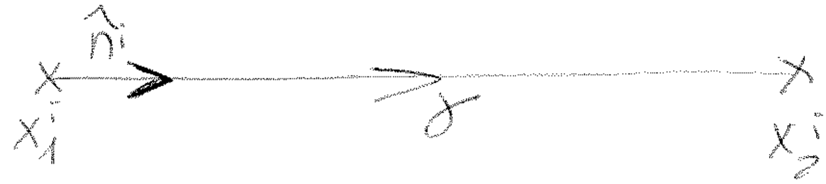
So, along photon geodesics:

$$\frac{d\Theta}{d\eta}=\psi'-\frac{d\phi}{d\eta}+\phi'$$

$$\Leftrightarrow \quad \boxed{\frac{d}{d\eta}(\Theta+\phi)=\frac{\partial}{\partial\eta}(\psi+\phi)}$$

Integral along line-of-sight

At order 1, we neglect the $\frac{d\hat{n}^i}{d\eta}$ term, so geodesics = straight line. Let us choose comoving coordinates such that:



with $\hat{n}^i = (1,0,0)$, so that:

$$x^i = \{x_1 + (\eta - \eta_1)\ ,\ 0,\ 0\}$$

(Remember that for photons $dx = d\eta$)

Integrating along the geodesics, we get:

$$\Theta(\eta_2, x_2^i, \hat{n}^i) + \phi(\eta_2, x_2^i) - \Theta(\eta_1, x_1^i, \hat{n}^i) - \phi(\eta_1, x_1^i)$$

$$= \int_{x_1}^{x_2} \frac{d}{dx}(\Theta + \phi)\, dx = \int_{\eta_1}^{\eta_2} \frac{d}{d\eta}(\Theta + \phi)\, d\eta$$

$$= \int_{\eta_1}^{\eta_2} \frac{\partial}{\partial \eta}(\psi + \phi)\, d\eta$$

In simpler notations: $\boxed{\Delta(\Theta + \phi) = \int_{\eta_1}^{\eta_2} (\psi' + \phi')\, d\eta}$

→ for static potentials: $\phi' = \psi' = 0$, and $\Theta + \phi$ is conserved along geodesics. $\frac{\delta T}{T}$ varies when photons cross gravitational potential wells, but the effect is conservative: $\Delta\Theta = -\Delta\phi$.

→ for non-static potentials, there is an integrated contribution: when photons travel through potential wells evolving with time, blueshift and redshift do not compensate each other...

<u>Note</u>: take static potentials. When Photons falls inside potential well:

$$\Delta\Phi < 0 \Leftrightarrow \Delta\Theta > 0 \Leftrightarrow \delta T \nearrow$$

Is this consistent? Inside potential well, g_{00} is smaller $\to$ frequencies are larger

$\to$ energies are smaller

$\to$ blackbody temperature is smaller

OK

Now, let x_1 be the ensemble of points on last scattering surface and x_2 be an observer of the CMB. In <u>instantaneous decoupling limit</u>, collisionless Boltzmann equation holds between x_1 and x_2. Moreover, if the observer looks in direction $\hat{n}$, he sees photons emitted in

$$x_1^i = r_{lss}\,\hat{n}^i$$

↳ radius of last scattering surface (lss):

$$r_{lss} = \underset{\text{today}}{\eta_0} - \underset{\text{decoupling}}{\eta_{dec}}$$

The position of the photon at time η is

$$x^i = \underbrace{(\eta_0 - \eta)}_{\geq 0}\,\hat{n}^i$$

The observer sees:

$$\frac{\delta T}{T}(\eta_0, x_2^i, \hat{n}^i) = \Theta(\eta_0, x_2^i, -\hat{n}^i)$$
$$= \Theta(\eta_{dec}, r_{lss}\hat{n}^i, -\hat{n}^i)$$
$$+ \Phi(\eta_{dec}, r_{lss}\hat{n}^i)$$
$$- \Phi(\eta_0, x_2^i)$$
$$+ \int_{\eta_{dec}}^{\eta_0} d\eta \left\{ \Phi'(\eta, (\eta_0-\eta)\hat{n}^i) + \Psi'(\eta, (\eta_0-\eta)\hat{n}^i) \right\}$$

= intrinsic temp. pert. on lss
+ local grav. pot. on lss
− local " " at the observer location
+ $(\Phi' + \Psi')$ integrated along line-of-sight

Conclusion: in each direction, the observer sees the same temperature anisotropy as on the l.s.s., corrected by:
* the redshift / blue shift experienced by the photons when they leave from a maximum / minimum of the gravitational potential on the l.s.s.
* the one experienced at the observer location (the observer may leave near a potential max / min)
* the integrated effect

Two remarks:

① we still don't know how to relate $\Theta(\eta_{dec}, r_{lss}\vec{n}, -\vec{n})$ to the quantities computed in Chapter IV.

This will become more clear in the next section $\boxed{\text{V-1b}}$.

(2) The correction $\phi(\eta_0, x^i_2)$ is the same in all directions of observation $\hat{n}^i$. So, it is a contribution to the monopole of $\frac{\delta T}{T}(\hat{n})$, or can be seen as an extra contribution to $\bar{T}$ (any monopole can be absorbed in a redefinition of $\bar{T}$:

$$T(\hat{n}) = \underbrace{\bar{T} - \bar{T}\phi_0}_{\tilde{T}} + \bar{T}\,\delta T(\hat{n})\ )$$

So, this ϕ is not detectable in practise. For simplicity we can set it to zero.

Fourier expansion

We define $\Theta(\eta, \vec{k}, \hat{n})$ through

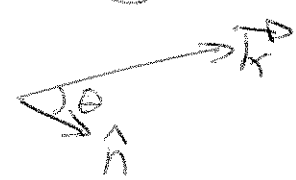
$$\Theta(\eta, \vec{x}, \hat{n}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\vec{x}}\, \Theta(\eta, \vec{k}, \hat{n})$$

Then Boltzmann $\Rightarrow$ $\Theta' + i\vec{k}\cdot\hat{n}\,\Theta = \Psi' - i\vec{k}\cdot\hat{n}\,\phi$ in Fourier space

Hence the equation depends on $\hat{n}$ only through the angle it forms with $\vec{k}$: $\vec{k}\cdot\hat{n} = k\cos\theta$

$$= k(\underset{\uparrow}{\hat{k}}\cdot\underset{\uparrow}{\hat{n}})$$

unit vectors



<u>Explanation</u>: Θ is fully symmetric under a rotation of $\hat{n}$ around the vector $\vec{k}$ (ie with constant θ).

It is in fact possible to show that this symmetry cannot be broken as long as the background is isotropic. Hence, in Fourier space, Θ is not a function of 6 variables, but only 5:

$$\Theta(\underbrace{\eta, \vec{k}, \hat{n}}_{1+3+2=6}) \longrightarrow \Theta(\underbrace{\eta, \vec{k}, \hat{k}\cdot\hat{n}}_{1+3+1=5}) \text{ only.}$$

Legendre expansion

In order to get convenient variables and equations, we want functions of only 4 variables: $(\eta, \vec{k})$. This is possible by doing a legendre expansion of the angle θ: we replace one function of θ by an infinity of multipoles not depending on θ:

$$\Theta(\eta, \vec{k}, \underbrace{\hat{k}\cdot\hat{n}}_{\cos\theta}) \equiv \sum_{\ell=0}^{\infty} (-i)^{\ell} (2\ell+1)\, \Theta_{\ell}(\eta, \vec{k})\; P_{\ell}(\hat{k}\cdot\hat{n})$$

Legendre momentum (or multipole)

Legendre polynomial

Basic properties of $P_\ell(X)$:

* $P_0(X) = 1 \quad P_1(X) = X$
* orthogonality: $\int \frac{d\hat{n}}{4\pi} P_{\ell}(\hat{k}\cdot\hat{n}) P_{\ell'}(\hat{k}\cdot\hat{n}) = \delta_{\ell\ell'} \frac{1}{2\ell+1}$

(In fact: $\hat{k}\,\hat{n} = \cos\theta$, $d\hat{n} = d\theta \sin\theta\, d\varphi$, and

$$\int \frac{d\hat{n}}{4\pi} P_\ell P_{\ell'} = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} \frac{d\varphi}{4\pi} P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \frac{1}{2}\int_0^{\pi} d\theta \sin\theta\, P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

$$= \frac{1}{2}\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \frac{1}{2}\left(\delta_{\ell\ell'} \frac{2}{2\ell+1}\right) = \frac{\delta_{\ell\ell'}}{2\ell+1} \quad)$$

* relation with spherical harmonics:

$$P_\ell(\hat{n}\cdot\hat{n}') = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{n})\, Y^*_{\ell m}(\hat{n}')$$

* inverse Legendre transformation: the orthogonality relation leads to

$$\int \frac{d\hat{n}}{4\pi}\, \Theta(\eta,\vec{k},\hat{k}\cdot\hat{n})\, P_\ell(\hat{k}\cdot\hat{n}) = (-i)^\ell\, \Theta_\ell(\eta,\vec{k})$$

* relation with plane waves: a plane wave can be expanded in Legendre multipoles using the identity:

$$e^{-i\vec{k}\cdot\vec{x}} = e^{-ikx(\hat{k}\cdot\hat{x})} = \sum_{\ell=0}^{\infty} (-i)^\ell (2\ell+1)\, \underbrace{j_\ell(kx)}\, P_\ell(\hat{k}\cdot\hat{x})$$

Legendre coefficient of $e^{i\vec{k}\vec{x}}$ is the spherical Bessel function, $j_\ell(x) \equiv \sqrt{\frac{\pi}{x}}\, J_{\ell+\frac{1}{2}}(x)$

usual Bessel function of the first kind.

We now expand the Boltzmann equation itself:

$$\sum_\ell (-i)^\ell (2\ell+1)\left[\Theta_\ell' P_\ell + ik(\hat{k}\cdot\hat{n})\,\Theta_\ell P_\ell\right] = \underbrace{1}_{P_0(\hat{k}\cdot\hat{n})} \times \psi' - ik\,\underbrace{(\hat{k}\cdot\hat{n})}_{P_1(\hat{k}\cdot\hat{n})}\,\phi$$

We can use the identity $(2\ell+1)X P_\ell(X) = (\ell+1)P_{\ell+1}(X) - \ell P_{\ell-1}(X)$ in order to express:

$$(\hat{k}\cdot\hat{n})P_\ell = \frac{\ell+1}{2\ell+1}P_{\ell+1} - \frac{\ell}{2\ell+1}P_{\ell-1}$$

Now, identifying each order in the Legendre expansion, we obtain:

$$\boxed{\begin{cases} \Theta_0' + k\,\Theta_1 = \Psi' \\ \Theta_1' - \frac{k}{3}\Theta_0 + \frac{2k}{3}\Theta_2 = \frac{k}{3}\phi \\ \forall \ell \geq 2 \quad \Theta_\ell' - k\frac{\ell}{2\ell+1}\Theta_{\ell-1} + k\frac{\ell+1}{2\ell+1}\Theta_{\ell+1} = 0 \end{cases}}$$

Physical meaning of first multipoles

For particles with phase-space distribution $f(x^i, P_j, \eta)$ and conjugate 4-momentum P_j, stress-energy tensor reads:

$$T_{\mu\nu} = \int dP_1 dP_2 dP_3 \sqrt{-g}\, \frac{P_\mu P_\nu}{P^0} f(x^i, P_j, \eta)$$

Using relations between P_0, P_j, p and $\hat{n}^j$ seen at beginning of this chapter, and replacing f by our ansatz, one obtains:

* $T^0_0 = \delta\rho_\gamma = \frac{\bar\rho_\gamma}{(-i)^\ell} \int \frac{d\hat{n}}{4\pi}\, 4\,\delta T(\eta, \vec{k}, \hat{n})$ in Fourier space

$$\Rightarrow \quad \delta_\gamma = \frac{1}{(-i)^\ell} \int \frac{d\hat{n}}{4\pi}\, 4\,\Theta(\eta, \vec{k}, \hat{k}\hat{n})$$

$$= \frac{4}{(-i)^\ell} \int \frac{d\hat{n}}{4\pi}\, P_0(\hat{k}.\hat{n})\,\Theta(\eta, \vec{k}, \hat{k}\hat{n}) \quad \text{since } P_0 = 1$$

$$= 4\,\Theta_0(\eta, \vec{k})$$

(inverse of Legendre transformation)

* similarily, after a few lines of calculation:

velocity divergence $\theta_\gamma = \partial_i T^i_0 = -3k\,\Theta_1$

anisotropic stress or shear $\sigma_\gamma = +2\,\Theta_2$

relation with continuity / Euler equations

Taking first two Boltzmann equations in Legendre space, and replacing $\Theta_{0,1,2}$ in terms of $\delta_\gamma, \theta_\gamma, \sigma_\gamma$:

$$\begin{cases} \delta_\gamma' - \frac{4}{3}\theta_\gamma = 4\psi' \\ \theta_\gamma' + \frac{1}{4}k^2\delta_\gamma - k^2\sigma_\gamma = -k^2\phi \end{cases}$$

In chapter II, we wrote the continuity / Euler equation for a species with $\frac{p}{\rho} = w = c_s^2 =$ constant and no coupling term:

$$\begin{cases} \delta_\gamma' = (1+w)(\theta_\gamma + 3\psi') \\ \theta_\gamma' = \frac{a'}{a}(3w-1)\theta_\gamma - k^2\phi - \frac{w}{1+w}k^2\delta_\gamma + k^2\sigma_\gamma \end{cases}$$

Taking $w = \frac{1}{3}$ (true for photons), we get

$$\begin{cases} \delta_\gamma' = \frac{4}{3}\theta_\gamma + 4\psi' \\ \theta_\gamma' = -k^2\phi - \frac{1}{4}k^2\delta_\gamma + k^2\sigma_\gamma \end{cases}$$

which is the same as above. Hence, the first two Legendre moments of the Boltzmann equation corresponds to the usual continuity equations.

Summary:
* for fluids, $\begin{cases} \text{2 variables: } \delta, \theta \\ \text{2 equations: continuity, Euler} \end{cases}$
* for free-streaming fluid (or, more generally, particles not necessarily strongly interacting),

$$\begin{cases} \infty \text{ variables: } \delta, \theta, \sigma, \Theta_3, \Theta_4, \dots \\ \infty \text{ equations: continuity, Euler + higher moments of Boltzmann} \end{cases}$$

V.1.b) Including the collision term:

The collision term accounting for Thomson scattering can be showed to be

$$C[f] = -p\frac{\partial f^0}{\partial p}\, a\, n_e\, \sigma_T \left[\Theta_0 - \Theta + \hat{n}\cdot\vec{v}_e\right]$$

with:
* $f^0(p,\eta)$ = homogeneous B.-E. distribution $\left(\frac{1}{e^{p/T(\eta)}-1}\right)$
* a = scale factor (present if we use conformal time: $\frac{df}{d\eta} = C[f]$ but $\frac{df}{dt} = \frac{1}{a}\frac{df}{d\eta} = \frac{C[f]}{a}$)
* n_e = number density of free electrons ($n_e = X_e\, n_e^{tot}$ with n_e^{tot} = density of electrons, X_e = ionisation fraction)
* σ_T = Thomson scattering rate
* $\Theta = \Theta(\eta, x^i, \hat{n}^i) = \frac{\delta T}{T}(\eta, x^i, \hat{n}^i)$
* $\Theta_0 = \int \frac{d\hat{n}}{4\pi}\,\Theta$ = monopole of Θ
* $\hat{n}$ = direction of momentum
* $\vec{v}_e$ = velocity of electrons (bulk velocity)
  $= \vec{v}_b$ because of Coulomb scattering

(Note that in this collision term, we could neglect Pauli blocking due to small occupation number of electrons after $e^+ + e^-$ annihilation, and stimulated emission which would contribute at the level of second order perturbations).

We saw in V.1.a. that

$$\frac{Df}{d\tau} = -p\frac{\partial f^0}{\partial p}\left[\Theta' + \hat{n}^i\partial_i\Theta - \psi' + n^i\partial_i\phi\right]$$

Hence the full Boltzmann equation reads:

$$\boxed{\Theta' + \hat{n}^i \partial_i \Theta - \varphi' + \hat{n}^i \partial_i \phi = a\, n_e \sigma_T \left(\Theta_0 - \Theta + \hat{n}^i v_{bi}\right)}$$

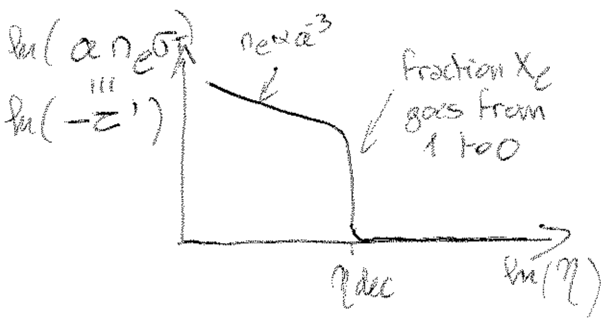
We define the optical depth: $\tau \equiv \int_\eta^{\eta_0} d\eta\; n_e \sigma_T a$

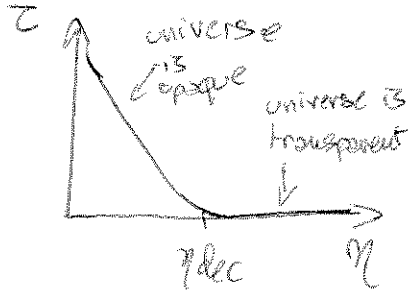
Interpretation: * τ is the scattering rate integrated along line of sight and normalized to $\tau(\eta_0) = 0$

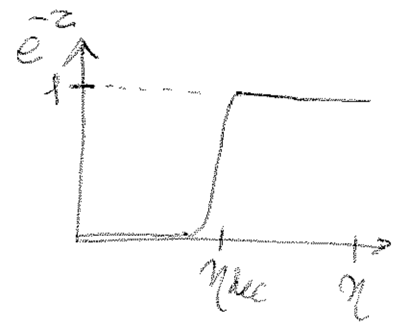
* $e^{-\tau(\eta)}$ = probability that photon emitted at η reaches us without scattering

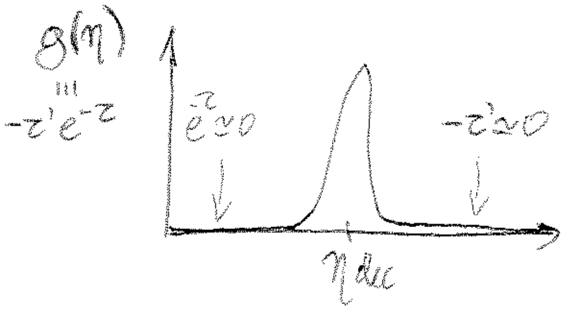
* $g(\eta) \equiv -\tau' e^{-\tau} \equiv$ visibility function: is the probability that a photon reaching us today <u>last scattered</u> at time η

Qualitative behavior of these function:









$\Rightarrow$ shows that most photons last scattered near decoupling time η_{dec}, width of $g(\eta)$ gives "duration of decoupling"...

The Boltzmann equation reads:

$$\boxed{\Theta' + \hat{n}^i \partial_i \Theta + \hat{n}^i \partial_i \phi - \Psi' = -\tau' \left(\Theta_0 - \Theta + \hat{n}^i v_{bi}\right)}$$

Integral along line-of-sight

Let us consider photons travelling along one given geodesics. We know that along the geodesics:

$$\frac{d}{d\eta}(\Theta + \Phi) = \Theta' + \Phi' + \hat{n}^i \partial_i \Theta + \hat{n}^i \partial_i \Phi$$

so that the full Boltzmann equation gives

$$\frac{d}{d\eta}(\Theta + \Phi) = \frac{\partial}{\partial \eta}(\Phi + \Psi) - \tau' \left[\Theta_0 - \Theta + \hat{n}\cdot\vec{v}_b\right]$$

→ we cannot readily integrate this equation due to term $-\tau'\Theta$ on right hand-side. In order to have a total derivative, we must multiply the equation by $e^{-\tau}$:

$$e^{-\tau}\frac{d}{d\eta}(\Theta + \Phi) + \tau' e^{-\tau}\Theta = e^{-\tau}\frac{\partial}{\partial \eta}(\Phi + \Psi) - \tau' e^{-\tau}\left[\Theta_0 + \hat{n}\cdot\vec{v}_b\right]$$

$$\Leftrightarrow \frac{d}{d\eta}\left[(\Theta + \Phi)e^{-\tau}\right] + \tau' e^{-\tau}\Phi = e^{-\tau}(\Phi' + \Psi') - \tau' e^{-\tau}\left[\Theta_0 + \hat{n}\cdot\vec{v}_b\right]$$

$$\Leftrightarrow \frac{d}{d\eta}\left[(\Theta + \Phi)e^{-\tau}\right] = e^{-\tau}(\Phi' + \Psi') + g(\eta)\left[\Theta_0 + \Phi + \hat{n}\cdot\vec{v}_b\right]$$

Let us consider a geodesics for photons passing at η_i through $\vec{x}_i$ in direction $\hat{n}_i$, and later at time η_f through $\vec{x}_f$ in direction $\hat{n}_f$. The integral along the geodesics over η gives:

$$e^{-\tau(\eta_f)}\left[\Theta(\eta_f, \vec{x}_f, \hat{n}_f) + \Phi(\eta_f, \vec{x}_f)\right] = \left[\Theta(\eta_i, \vec{x}_i, \hat{n}_i) + \Phi(\eta_i, \vec{x}_i)\right] e^{-\tau(\eta_i)} + \int_{\eta_i}^{\eta_f} d\eta \left\{ e^{-\tau}(\Phi' + \Psi') + g(\eta)\left[\Theta_0 + \Phi + \hat{n}\cdot\vec{v}_b\right]\right\}$$

Let us take $\eta_f = \eta_0 =$ today ($\Rightarrow e^{-\tau(\eta_f)} = 1$) and $\eta_i \to -\infty$, $\eta_i \ll \eta_{dec}$ ($\Rightarrow e^{-\tau(\eta_i)} \to 0$). Then:

$$\Theta(\eta_0, \vec{x}_0, \hat{n}_0) + \Phi(\eta_0, \vec{x}_0, \hat{n}_0) = \int_0^{\eta_0} d\eta \left\{ e^{-\tau}(\Phi' + \Psi') + g(\eta)\left[\Theta_0 + \Phi + \hat{n}\cdot\vec{v}_b\right]\right\}$$

Now, let us work in the instantaneous decoupling approximation. In this limit:

* $g(\eta) \propto \delta(\eta - \eta_{dec})$. Actually the correct normalization is $g(\eta) = \delta(\eta - \eta_{dec})$ because $\int_0^{\eta_0} g(\eta)\, d\eta = \int_0^{\eta_0} -\tau' e^{-\tau} d\eta = \left[e^{-\tau} \right]_0^{\eta_0} = 1$

* $e^{-\tau} = H(\eta - \eta_{dec})$ because $\tau(\eta \leq \eta_{dec}) = \infty$, $\tau(\eta \geq \eta_{dec}) = 0$
* geodesics = straight line between η_{dec} and η_0 (as in discussion of section V.1.a.), $\vec{x} = -(\eta_0 - \eta)\, \hat{n}$

So:

(at decoupling: $\eta_0 - \eta_{dec} \equiv r_{lss}$) radius of last scattering surface

$$\begin{aligned} \Theta(\eta_0, \vec{x}_0, \hat{n}) = & -\phi(\eta_0, \vec{x}_0) \\ & + \Theta_0(\eta_{dec}, -r_{lss}\hat{n}, \hat{n}) \\ & + \phi(\eta_{dec}, -r_{lss}\hat{n}) + \hat{n} \cdot \vec{v}_b(\eta_{dec}, -r_{lss}\hat{n}) \\ & + \int_{\eta_{dec}}^{\eta_0} d\eta\, (\phi' + \psi')\Big|_{\eta, -(\eta_0 - \eta)\hat{n}} \end{aligned}$$

A CMB observer located at $\vec{x}_0$ today sees a temperature anisotropy

$$\begin{aligned} \frac{\delta T}{T}(\eta_0, \vec{x}_0, \hat{n}) &= \Theta(\eta_0, \vec{x}_0, -\hat{n}) \\ &= -\phi(\eta_0, \vec{x}_0) \\ &\quad + \left[\Theta_0 + \phi + \hat{n} \cdot \vec{v}_b \right]\Big|_{\eta_{dec}, r_{lss}\hat{n}, -\hat{n}} \\ &\quad + \int_{\eta_{dec}}^{\eta_0} d\eta\, (\phi' + \psi')\Big|_{\eta, (\eta_0 - \eta)\hat{n}} \end{aligned}$$

(direction of observation $= \hat{n}$, orientation of geodesics $= -\hat{n}$ ⚠)

We obtain the same formula as in VI-a expected that, since we included the collision term, $\Theta(\eta_{dec}, r_{lss}\vec{n}, -\hat{n})$ is replaced by $(\Theta_0 + \hat{n}\cdot\vec{v}_b)$! We can now interpret each term:

$$\frac{\delta T}{T}\Big|_0 = \underset{①}{-\phi|_{obs}} + \underset{②}{\Theta_0|_{lss}} + \underset{③}{\phi|_{lss}} + \underset{④}{\hat{n}\cdot\vec{v}_b|_{lss}} + \underset{⑤}{\int_{\eta_{dec}}^{\eta_0} d\eta\,(\phi'+\psi')}$$

① = contribution to monopole: not detectable, absorbed in definition of background temperature

② = intrinsic $\frac{\delta T}{T}$ on l.s.s.

③ = gravitational redshift due to inhomogeneities on lss

④ = Doppler effect due to bulk velocity of e^-, b on l.s.s.

⑤ = integrated Sachs-Wolfe effect along line of sight

② + ③ is called the Sachs-Wolfe effect.

Sachs-Wolfe formula

The above formula, valid only in the instantaneous decoupling limit, is called the Sachs-Wolfe approximation to observable CMB anisotropies. Neglecting the monopole contribution $\phi|_0$:

$$\frac{\delta T}{T}\Big|_0 = \underbrace{\Theta_0|_{lss} + \phi|_{lss}}_{\text{Sachs-Wolfe}} + \underbrace{\hat{n}\cdot\vec{v}_b|_{lss}}_{\text{Doppler}} + \underbrace{\int_{\eta_{dec}}^{\eta_0} d\eta\,(\phi'+\psi')}_{\text{Integrated Sachs Wolfe (ISW)}}$$

It is easy to compute the Sachs-Wolfe term as a function of Φ_{lss} only, using results of Chapter IV. Decoupling takes place during matter domination, when $\delta_m = -2\Phi$ and $\delta_m = \frac{3}{4}\delta_r$ for modes $k \ll aH$.

Hence $\delta_r = -\frac{8}{3}\Phi$, with $\delta_r = \delta_\gamma = 4\frac{\delta T}{T} = 4\Theta_0$.

So $\Theta_0 = \frac{1}{4}\delta_r = -\frac{2}{3}\Phi$, and for $k \ll aH$, $\Theta_0 + \Phi = \frac{1}{3}\Phi$.

The Sachs-Wolfe formula is true in real space, not Fourier space. However, if we are only interested in large-angle correlations in CMB maps (for angles larger than that of Hubble radius at decoupling), we can do as if $\Theta_0 + \Phi = \frac{1}{3}\Phi$ applied in real space, instead of just large wavelength Fourier modes. Moreover, for these scales we can neglect baryon velocities, and:

$$\left(\frac{\delta T}{T}\right)^{\text{observed}}(\hat{n}) \simeq \underbrace{\frac{1}{3}\Phi(\hat{n}\, r_{lss})}_{\substack{\text{Sachs-Wolfe} \\ \text{(adiabatic mode)}}} + \underbrace{\int_{\eta_{dec}}^{\eta_0} (\Phi' + \Psi')\, d\eta}_{\text{ISW}}$$

This crude approximation doesn't make sense for understanding fine-structure of CMB maps, and still assumes instantaneous decoupling. Note that $\frac{\delta T}{T} > 0$ $\Leftrightarrow \Phi > 0$: $\begin{cases} \text{hot spots} \Leftrightarrow \text{underdense regions !!!} \\ \text{cold spots} \Leftrightarrow \text{overdense regions !!!} \end{cases}$

Interpretation: photons leaving from overdense region must climb out of potential well, getting redshifted and loosing energy: they will produce a cold spot!

Fourier and Legendre expansion of full Boltzmann equation:

We perform the same transformations as in V-1a, including now collision terms:

Fourier → $\Theta(\eta, \vec{k}, \hat{k}\hat{n})$ with $\hat{k}\cdot\hat{n} = \cos\theta \equiv \mu$ obeys to:

$$\Theta' + i\underbrace{\vec{k}\cdot\hat{n}}_{k\mu}\Theta - \Psi' + i\underbrace{\vec{k}\cdot\hat{n}}_{k\mu}\phi = -\tau'\left(\Theta_0 - \Theta + i\underbrace{\vec{k}\cdot\hat{n}}_{k\mu}\tilde{v}_b\right)$$

where we defined $\vec{v}_b = \partial_i \tilde{v}_b$ (the curl component of $\vec{v}_b$ does not couple with scalar perturbations and with Θ). This $\tilde{v}_b$ is related to the quantity $\theta_b = \partial_i T^i_0$ of Chapter IV through $\tilde{v}_b = k^{-2}\theta_b$. Hence:

$$\boxed{\Theta' + ik\mu\Theta - \Psi' + ik\mu\phi = -\tau'\left(\Theta_0 - \Theta + ik^{-1}\mu\,\theta_b\right)}$$

Legendre → $$\sum_\ell (-i)^\ell(2\ell+1)\left[\Theta_\ell' + ik\mu\Theta_\ell - \tau'\Theta_\ell\right]P_\ell(\mu)$$
$$= P_0(\mu)\Psi' - ikP_1(\mu)\phi - \tau' P_0(\mu)\Theta_0 - \tau' i k^{-1} P_1(\mu)\theta_b$$

Using $\mu P_\ell(\mu) = \frac{\ell+1}{2\ell+1}P_{\ell+1}(\mu) - \frac{\ell}{2\ell+1}P_{\ell-1}(\mu)$ and identifying each order's coefficient, we obtain:

$$\boxed{\begin{aligned}
&\Theta_0' + k\Theta_1 = \Psi' \\
&\Theta_1' - \frac{k}{3}\Theta_0 + \frac{2k}{3}\Theta_2 = \frac{k}{3}\phi + \tau'\left(\frac{1}{3k}\theta_b + \Theta_1\right) \\
\forall \ell \geq 2\quad &\Theta_\ell' - k\frac{\ell}{2\ell+1}\Theta_{\ell-1} + k\frac{\ell+1}{2\ell+1}\Theta_{\ell+1} = \tau'\Theta_\ell
\end{aligned}}$$

As in V.1.2., we can identify Θ_0 with $\frac{\delta_\gamma}{4}$, Θ_1 with $-\frac{1}{3k}\theta_\gamma$ and Θ_2 with $\frac{1}{2}\sigma_\gamma$. The first two equations become:

$$\begin{cases} \delta_\gamma' - \frac{4}{3}\theta_\gamma = 4\psi' & \text{Continuity} \\ \theta_\gamma' + \frac{1}{4}k^2\delta_\gamma - k^2\sigma_\gamma = -k^2\Phi + \underbrace{\tau'(\theta_\gamma - \theta_b)}_{\text{new coupling term with baryons (through electrons)!}} & \text{Euler} \end{cases}$$

Remarks

① In the decoupled limit $\tau' \to 0$, valid after η_{dec} and called "free-streaming limit", the system formed by continuity + Euler equations is not closed. Need ∞ of equations! As time passes by, higher and higher multipoles l get populated (since Θ_ℓ' depends on $\Theta_{\ell+1}$)

② In the tight-coupling limit $\tau' \to \infty$, valid for $\eta \ll \eta_{dec}$, the right-hand side of Boltzmann in Fourier space reads : $\infty\left[\Theta_0 - \Theta + ik^{-1}P_1(\mu)\tilde{\theta}_b\right]$

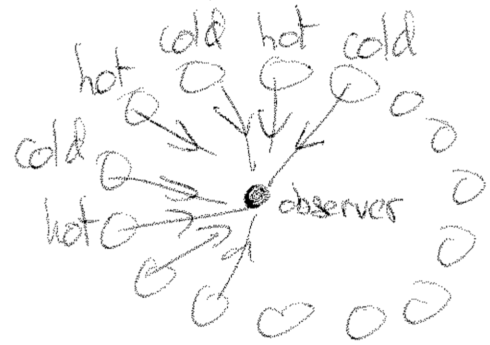
So $\Theta_0 - \Theta + ik^{-1}P_1(\mu)\theta_b \longrightarrow 0$, implying that:

- $\Theta_{\ell\geq 2}$ vanish, including $\Theta_2 = \frac{\sigma_\gamma}{2}$. Hence, we have a fluid.
- $3i\Theta_1 = -ik^{-1}\theta_b \iff \theta_\gamma = \theta_b$: Hence, Θ has a <u>dipole</u> component <u>imposed</u> by the bulk velocity of electrons/baryons. In other words: in the referential comoving with the electrons,

$\frac{\delta T}{T}$ would be <u>isotropic</u> like for true Bose-Einstein distribution with local temperature $T+\delta T$. In other frames or gauges, the temperature gets a dipole (just like the observed CMB temperature gets a dipole from the peculiar velocity of the earth!)

<u>Summary</u>: * for $\eta \ll \eta_{dec}$,
$$\Theta = \Theta_0 - 3i\,\Theta_1 P_1(\mu) = \underbrace{\Theta_0}_{\substack{\text{monopole} \\ \parallel \\ \frac{1}{4}\delta_\gamma}} \underbrace{- i k^{-1} \mu\, \Theta_b}_{\substack{\text{dipole} \\ \parallel \\ -ik^{-1}\mu\,\Theta_\gamma}}$$

* after η_{dec}, Θ_0 and Θ_1 couple with higher momenta $\Theta_{\ell \geq 2}$; higher ℓ's get progressively populated (meaning that observer sees anisotropies on smaller and smaller angular scale, due to the superposition of many flows in the same point)



radius of lss increases with time for given observer
$\rightarrow$ he can see more and more structures!