V.3. From perturbations to CMB anisotropies

Our description of cosmological perturbations is complete now, but not directly usable: many observations (in particular, of the CMB) consist in two-dimensional maps of the sky, not 3D maps that could be expanded in Fourier space...

$$\text{CMB map} \equiv \frac{\delta T}{T}(\vec{n}) = \sum_{\ell m} \underbrace{a_{\ell m}}_{\substack{\uparrow \\ \text{complex number} \\ \text{with } a_{\ell\,-m} = (-1)^m a^*_{\ell m} \ \left(\frac{\delta T}{T} \text{ being real}\right)}} \overbrace{Y_{\ell m}}^{\text{spherical harmonics}}(\vec{n})$$

In the following, we will find relations between $a_{\ell m}$ and perturbations in Fourier space $\left(\Theta_\ell(\eta, \vec{k})\right)$.

We will prove that:

* $a_{\ell m}$'s are stochastic, Gaussian, with zero mean value (like each Fourier mode $\delta_i(\eta, \vec{k})$)

* $a_{\ell m}$'s are independent of eachother: $\langle a_{\ell m} a^*_{\ell' m'} \rangle \propto \delta_{\ell\ell'} \delta_{mm'}$ (like Fourier modes: $\langle \delta_i(\eta, \vec{k}) \delta_i^*(\eta, \vec{k}') \rangle \propto \delta^{(3)}(\vec{k} - \vec{k}')$)
* two-point correlation function depends on ℓ only due to isotropy: $\langle a_{\ell m} a^*_{\ell' m'} \rangle = \underset{\substack{\uparrow \\ \text{power spectrum} \\ \text{in harmonic space}}}{C_\ell} \delta_{\ell\ell'} \delta_{mm'}$

(similarily to Fourier space: $\langle \delta_i(\eta, \vec{k}) \delta_i^*(\eta, \vec{k}') \rangle = \delta^{(3)}(\vec{k} - \vec{k}') \times \underbrace{\langle |\delta_i(\eta, \vec{k})|^2 \rangle}_{\frac{2\pi^2}{k^3} \mathcal{P}_{\delta_i}(k)}$)

Hence, all information on CMB temperature anisotropies for one given cosmological model is contained in the power spectrum in harmonic space: C_ℓ !!!

The cosmological perturbation theory predicts that the two-point correlation function of the map should be given by:

$$\left\langle \frac{\delta T}{T}(\hat{n}) \frac{\delta T}{T}(\hat{n}') \right\rangle = \sum_{\substack{\ell m \\ \ell' m'}} \langle a_{\ell m} a^*_{\ell' m'} \rangle Y_{\ell m}(\hat{n}) Y_{\ell' m'}(\hat{n}')$$

$$= \sum_{\ell m \, \ell' m'} C_\ell \, \delta_{\ell \ell'} \delta_{m m'} Y_{\ell m}(\hat{n}) Y_{\ell' m'}(\hat{n}')$$

$$= \sum_\ell C_\ell \underbrace{\sum_m Y_{\ell m}(\hat{n}) Y^*_{\ell m}(\hat{n}')}$$

$$= \sum_\ell C_\ell \left(\frac{2\ell+1}{4\pi} P_\ell(\hat{n}.\hat{n}') \right)$$

This is a theoretical average, on many possible realization of the stochastic theory. Observers cannot do such an average... but they can compute the average over all possible $\hat{n}$ and $\hat{n}'$ with fixed $\hat{n}.\hat{n}' = \cos\theta$. By ergodicity, the second should tend to the first (in the limit of many independent $\hat{n}$'s and $\hat{n}'$'s)

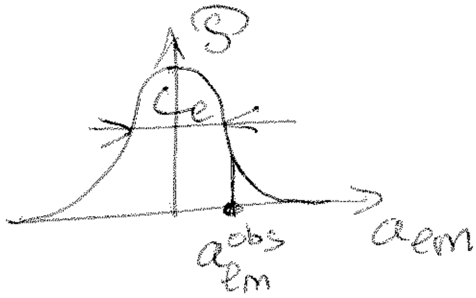
Hence, if we know C_ℓ, we can predict the value of the 2-point correlation function in real space.

Cosmic variance

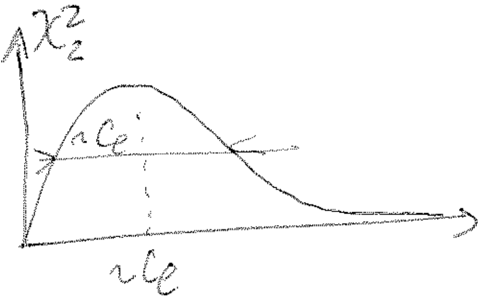
In the theory, $C_\ell = \langle a_{\ell m} a^*_{\ell m} \rangle$ → theoretical average over all possible realizations of the stochastic theory.

In real life, we see only one realization and the measurement of a single $a_{\ell m}$ does not provide C_ℓ !! However we can buid "estimators" allowing to measure C_ℓ up to "cosmic variance". Indeed, suppose that we measure $\frac{\delta T}{T}^{obs}(\hat{n})$ and expand it in $a^{obs}_{\ell m}$.

Each $a_{\ell m}^{obs}$ is a realization of a Gaussian probability distribution of a Gaussian with variance C_ℓ:



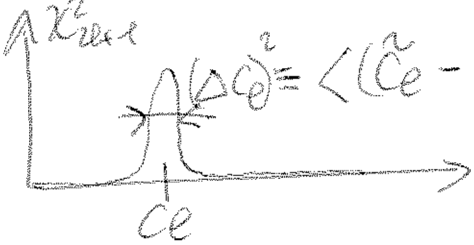
The product $a_{\ell m}^{obs} a_{\ell m}^{obs\,*}$ is an estimator of C_ℓ (ie, its typical value would be C_ℓ), but with a large dispersion. Indeed, the product $a_{\ell m} a_{\ell m}^*$ should obey a χ^2 distribution with 2 d.o.f., for whic the variance is of order C_ℓ:



Since all $a_{\ell m}$'s with <u>fixed ℓ</u> obey to the same probability, we can buid the average quantity:

$$\tilde{C}_\ell \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}^{obs}|^2$$

Again this is an estimator of C_ℓ, which is much better than $|a_{\ell m}^{obs}|^2$ alone, since it obeys to a χ^2 distribution with $2\ell+1$ d.o.f.:



$$(\Delta C_\ell)^2 = \langle (\tilde{C}_\ell - C_\ell)^2 \rangle = \frac{2}{2\ell+1} C_\ell^2$$

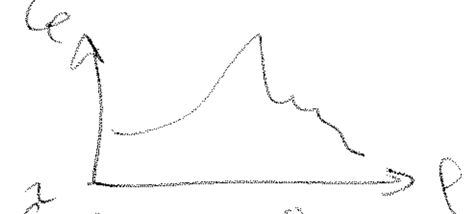
Indeed, in $\tilde{C}_\ell$ we average over $2\ell+1$ independent realizations of the same Gaussian (indeed: since $a_{\ell m} = a^*_{\ell m}$),

- in $a_{\ell 0} \rightarrow 1$ independent realization
- in $a_{\ell m}$ with $1 \le m \le \ell$: 2ℓ independent realizations (real + imaginary part)
- in $a_{\ell m}$ with $-\ell \le m \le -1$: no further independent realization)

So, $\tilde{C}_\ell$ is a good estimator of C_ℓ: the measured $\tilde{C}_\ell$ should be such that $(\tilde{C}_\ell - C_\ell)^2 \le \frac{2}{2\ell+1} C_\ell^2$ in 68% of the cases.

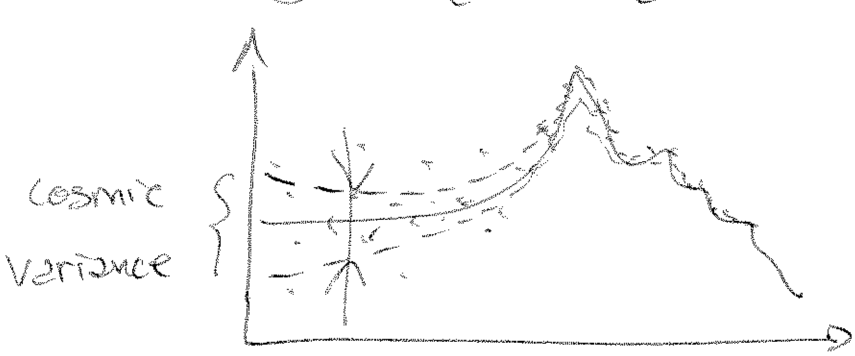
The corresponding errorbar on C_ℓ, $\Delta C_\ell = \sqrt{\frac{2}{2\ell+1}}\, C_\ell$, is called "cosmic variance" or "sampling variance".

A theory predicts C_ℓ's:



An observation provides $\tilde{C}_\ell$'s:

The theory is valid if 68% of $\tilde{C}_\ell$'s are in the range $C_\ell \pm \Delta C_\ell$:



When $\ell \nearrow$: number of $m \nearrow \Rightarrow \Delta C_\ell \searrow \Rightarrow$ cosmic variance becomes very small

Relation between harmonic space ($a_{\ell m}$'s) and multipoles in Fourier space ($\Theta_\ell(\eta, \vec{k})$):

By definition, $$\frac{\delta T}{T}^{obs}(\underbrace{\hat{n}}_{\text{direction of observation}}) = \Theta(\overset{\text{today}}{\eta_0}, \overset{\text{location of observer}}{\vec{x}_0}, \underbrace{-\hat{n}}_{\text{direction of propagation of photons coming from direction } \hat{n}}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n})$$

So $$\Theta(\eta_0, \vec{x}_0, \hat{n}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(-\hat{n})$$

If $\hat{n} = \underbrace{\begin{pmatrix}\theta \\ \varphi\end{pmatrix}}_{\text{spherical coordinates}}$, $-\hat{n} = \begin{pmatrix}\pi-\theta \\ \varphi+\pi\end{pmatrix}$; but $Y_{\ell m}(\pi-\theta, \varphi+\pi) = (-1)^\ell Y_{\ell m}(\theta, \varphi)$

so $Y_{\ell m}(-\hat{n}) = (-1)^\ell Y_{\ell m}(\hat{n})$

So $$\Theta(\eta_0, \vec{x}_0, \hat{n}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}) (-1)^\ell$$

Using the orthogonality relation:

$$\int d\hat{n}\, Y_{\ell m}(\hat{n}) Y_{\ell' m'}(\hat{n}) = \delta_{\ell\ell'}\delta_{mm'}$$ (Note: $d\hat{n} = d\Omega = d\theta \sin\theta\, d\varphi$)

we can extract the coefficient $a_{\ell m}$:

$$a_{\ell m} = \int d\hat{n}\, (-1)^\ell\, \Theta(\eta_0, \vec{x}_0, \hat{n}) Y^*_{\ell m}(\hat{n})$$

Fourier expansion of Θ:

$$a_{\ell m} = (-1)^\ell \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \int d\hat{n}\, \Theta(\eta_0, \vec{k}, \hat{k}\cdot\hat{n}) Y^*_{\ell m}(\hat{n})$$

Legendre expansion of Θ:

$$a_{\ell m} = (-1)^\ell \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}_0} \sum_{\ell'} (-i)^{\ell'}(2\ell'+1)\Theta_{\ell'}(\eta_0, \vec{k}) \int d\hat{n}\, P_{\ell'}(\hat{k}\cdot\hat{n}) Y^{m*}_{\ell}(\hat{n})$$

We expand $$P_{\ell'}(\hat{k}\cdot\hat{n}) = \sum_{m'} Y_{\ell' m'}(\hat{k}) Y_{\ell' m'}(\hat{n}) \frac{4\pi}{2\ell'+1}$$

Carrying integral on $d\hat{n}$ and using orthogonality of $Y_{\ell' m'}$'s:

$$a_{\ell m} = (-1)^\ell \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\vec{x}_0} \sum_{\ell'} (-i)^{\ell'} (2\ell'+1) \Theta_{\ell'} \sum_{m'} Y_{\ell' m'}(\hat{k}) \frac{4\pi}{2\ell'+1} \delta_{\ell\ell'} \delta_{mm'}$$

$$= (-1)^\ell \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\vec{x}_0} (-i)^\ell \cancel{(2\ell+1)} \Theta_\ell Y_{\ell m}(\hat{k}) \frac{4\pi}{\cancel{(2\ell+1)}}$$

Without loss of generality, and for simplicity, we can put the origin at the observer's location: $\vec{x}_0 = \vec{0}$. Then:

$$\boxed{a_{\ell m} = (i)^\ell \int \frac{d^3\vec{k}}{2\pi^2} \Theta_\ell(\eta_0, \vec{k}) Y_{\ell m}(\hat{k})}$$

It is useful to remember that $\Theta_\ell(\eta, \vec{k})$ is the product of an initial condition (= stochastic number depending on $\vec{k}$) and a transfer function accounting for the evolution btw early universe and today (= function depending on η and k, not $\vec{k}$ because of isotropy).

For instance, for the adiabatic mode, I.C. can be expressed as $\mathcal{R}(\vec{k})$ = curvature perturbation for mode $\vec{k}$ when it is outside the Hubble radius ($k \ll aH$) and constant in time. So:

$$\Theta_\ell(\eta, \vec{k}) = \underbrace{\Delta_\ell(\eta, k)}_{\substack{\text{transfer functions of} \\ \text{photons, defined with respect to } \mathcal{R}}} \mathcal{R}(\vec{k})$$

Then:

$$\boxed{a_{\ell m} = (i)^\ell \int \frac{d^3\vec{k}}{2\pi^2} \Delta_\ell(\eta_0, k) Y_{\ell m}(\hat{k}) \mathcal{R}(\vec{k})}$$

Relation between anisotropy power spectrum (C_ℓ) and Fourier power spectrum ($\langle|\Theta_\ell|^2\rangle$ or $\Delta_\ell^2\langle|\mathcal{R}|^2\rangle$)

We can compute

$$\langle a_{\ell m} a^*_{\ell' m'}\rangle = (i)^{\ell-\ell'} \int \frac{d^3\vec{k}}{(2\pi^3)} \int \frac{d^3\vec{k}'}{(2\pi^3)} \Delta_\ell \Delta^*_{\ell'} Y_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{k}') \underbrace{\langle \mathcal{R}(\vec{k}) \mathcal{R}^*(\vec{k}')\rangle}_{\substack{\| \\ \delta^{(3)}(\vec{k}-\vec{k}')\langle|\mathcal{R}(\vec{k})|^2\rangle \\ \| \\ \delta^{(3)}(\vec{k}-\vec{k}')\frac{2\pi^2}{k^3}\mathcal{P}_\mathcal{R}(k)}}$$

$$= (-i)^{\ell-\ell'} \int \frac{d^3\vec{k}}{2\pi^2} k^{-3} \Delta_\ell(\eta_0,k)\Delta_{\ell'}(\eta_0,k) Y_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{k}) \mathcal{P}_\mathcal{R}(k)$$

But $d^3\vec{k} = dk\, k^2 d\theta \sin\theta\, d\varphi = dk\, k^2\, d\hat{k}$

$$\langle a_{\ell m} a^*_{\ell' m'}\rangle = (-i)^{\ell-\ell'} \int \frac{dk}{2\pi^2 k} \Delta_\ell(\eta_0,k)\Delta_{\ell'}(\eta_0,k)\mathcal{P}_\mathcal{R}(k) \underbrace{\int d\hat{k}\, Y_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{k})}_{\delta_{\ell\ell'}\delta_{mm'}}$$

So:

$$\langle a_{\ell m} a^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}\left[\frac{1}{2\pi^2}\int \frac{dk}{k} |\Delta_\ell(\eta_0,k)|^2 \mathcal{P}_\mathcal{R}(k)\right]$$

Hence we can identify

$$\boxed{\begin{aligned} C_\ell &= \frac{1}{2\pi^2}\int \frac{dk}{k} |\Delta_\ell(\eta_0,k)|^2 \mathcal{P}_\mathcal{R}(k) \\ &= \frac{1}{4\pi^4}\int k^2 dk\, |\Delta_\ell(\eta_0,k)|^2 \langle|\mathcal{R}(\vec{k})|^2\rangle \end{aligned}}$$

(Note: some authors define Fourier transform as

$\delta_x = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \cdots$ instead of $\int \frac{d^3\vec{k}}{(2\pi)^3} \cdots$

In the expression of C_ℓ, they get the same result as us multiplied by $(2\pi)^3$!)

Brute force calculation of C_ℓ's :

At this stage, we know all formulas and equations necessary for the C_ℓ's. A brute force calculation would consist in running a code doing the following:

* the code should know the full system of equations in Fourier space:

  Continuity + Euler for δ_c, θ_c
  " " " δ_b, θ_b
  Boltzmann for $\delta_\gamma, \theta_\gamma, \sigma_\gamma, \Theta_{\ell \geq 3}$
  (+ Einstein used as constraint equations providing metric perturbations)

* the code integrates this system from η_{ini} to η_0, starting from initial condition: $\forall \vec{k}$, $\mathcal{R}(\vec{k}) = 1$, so that $\Theta_\ell(\eta, \vec{k})$ of the code corresponds to $\Delta_\ell(\eta, k)$ of this course. Hence, as a result, we get $\Delta_\ell(\eta_0, k)$ for all ℓ's and k's.
* the code performs finally the convolution with the primordial spectrum:

$$C_\ell = \frac{1}{2\pi^2} \int \frac{dk}{k} |\Delta_\ell(\eta_0, k)|^2 \mathcal{P}_{\mathcal{R}}(k)$$

THIS APPROACH TAKES OF THE ORDER OF A FEW DAYS for on spectrum C_ℓ.

The computation time is large due to:

(a) Number of variables $\Delta_\ell(\eta, k)$

need many ℓ's in order to get C_ℓ's up to $l_{max} \sim 2500$.

need several k's in order to sample all acoustic oscillations in observable range of wavelengths

Typically, the Boltzmann hierarchy can only be truncated around $\sim 2 l_{max}$

So, typically, one needs ~ 5000 ℓ's and $\sim 10^3$ k's...

(b) Need for small time-step before decoupling (interaction time very small for photons). However this can be alleviated by using tight-coupling approximation for $\eta \ll \eta_{dec}$

(c) Need for small time-step after decoupling, to follow the way in which high ℓ's are populated during the free-streaming epoch (starting from just Θ_0 and Θ_ℓ being non-zero).

INTUITIVELY, WE ~~SEE~~ THAT THERE SHOULD BE A MORE CLEVER APPROACH BECAUSE:

Population of high l's after decoupling should be universal (free-streaming does not depend on cosmological parameters)

$\Downarrow$

should be a waste to integrate Θ_ℓ's for high ℓ's, only first few ℓ's encode non-trivial evolution

$\Rightarrow$ This motivate the next sub-section

line-of-sight integral in Fourier space (!!!)

We want to do as in real space: identify total derivative over time and integrate....

$\mu \equiv \hat{k}.\hat{n}$

Boltzmann $\rightarrow$ $\Theta' + ik\mu\Theta - \Psi' + ik\mu\Phi = -\tau'(\Theta_0 - \Theta + i\frac{\mu}{k}\theta_b)$

$\Leftrightarrow \Theta' + (ik\mu - \tau')\Theta = \Psi' - ik\mu\Phi - \tau'(\Theta_0 + i\frac{\mu}{k}\theta_b)$

$\Leftrightarrow \frac{d}{d\eta}\left[\Theta e^{ik\mu\eta - \tau}\right] = \left[\Psi' - ik\mu\Phi - \tau'(\Theta_0 + i\frac{\mu}{k}\theta_b)\right]e^{ik\mu\eta - \tau}$

We integrate from η_{ini} to η_0 (today). We can use the fact that $\tau(\eta_0)=0$. Also, we take the limit $\eta_{ini} \rightarrow 0$, so that $e^{-\tau(\eta_{ini})} \longrightarrow 0$. Then:

$$\Theta(\eta_0, \vec{k}, \mu)\, e^{ik\mu\eta_0} = \int_0^{\eta_0} d\eta \left[\Psi' - ik\mu\Phi - \tau'(\Theta_0 + i\frac{\mu}{k}\theta_b)\right]e^{ik\mu\eta - \tau}$$

$$\Leftrightarrow \Theta(\eta_0, \vec{k}, \mu) = \int_0^{\eta_0} d\eta \left[\Psi' - ik\mu\Phi - \tau'(\Theta_0 + i\frac{\mu}{k}\theta_b)\right] e^{ik\mu(\eta - \eta_0) - \tau}$$

It is usefull to integrate by part in order to eliminate all dependence on μ inside the brackets:

$$\int_0^{\eta_0} d\eta\, f(k,\eta)\, ik\mu\, e^{ik\mu(\eta-\eta_0) - \tau}$$

$$= -\int_0^{\eta_0} d\eta \left(f(k,\eta)e^{-\tau}\right)' e^{ik\mu(\eta - \eta_0)} + \left[f(k,\eta)e^{-\tau}e^{ik\mu(\eta-\eta_0)}\right]_0^{\eta_0}$$

* vanishes in η_0 due to $e^{-\tau(0)} = 0$
* in η_0, contribution only to monopole: $f(k,\eta_0)$ does not depend on μ. Hence, undetectable (absorbed in background T).

We apply this formula with $f(k,\eta) = -\Phi$ or $-\frac{\tau'}{k^2}\theta_b$ and get:

$$\Theta(\eta_0, \vec{k}, \mu) = \int_0^{\eta_0} d\eta \left[\psi' e^{-\tau} + (\phi e^{-\tau})' - \tau' e^{-\tau} \Theta_0 + \left(\frac{\tau'}{k^2} \theta_b e^{-\tau} \right)' \right] e^{ik\mu(\eta - \eta_0)}$$
$$= \int_0^{\eta_0} d\eta \; \tilde{S}(\vec{k}, \eta) e^{ik\mu(\eta - \eta_0)}$$

with $\tilde{S}(\vec{k}, \eta) = (\psi' + \phi')e^{-\tau} + \underset{-\tau' e^{-\tau}}{\underset{|||}{g(\eta)}} (\phi + \Theta_0) + \left(g(\eta) \frac{\theta_b}{k^2} \right)'$

where $\psi', \phi', \phi, \Theta_0$ and θ_b are functions of $\vec{k}, \eta$.

This expression can easily be expanded in Legendre coefficients, because the plane wave $e^{i\vec{k}\vec{x}}$ has simple Legendre coefficients: the spherical Bessel function $j_\ell(X) = \sqrt{\frac{\pi}{2X}} J_{\ell+1}(X)$

↑ usual Bessel function of 1st kind.

Take $\vec{x} = (\eta_0 - \eta)\hat{n}$ = position of a photon travelling in straight line and seen today in direction $\hat{n}$.

Then $e^{-i\vec{k}\vec{x}} = \sum_{\ell=0}^{\infty} (-i)^\ell (2\ell+1) j_\ell(kx) P_\ell(\hat{k}\hat{x})$

$$\Leftrightarrow e^{-ik(\eta_0 - \eta)\mu} = \sum_{\ell=0}^{\infty} (-i)^\ell (2\ell+1) j_\ell(k(\eta_0 - \eta)) P_\ell(\mu)$$

So, we can expand $\Theta = \int d\eta \, \tilde{S} e^{ik\mu(\eta - \eta_0)}$ in Legendre multipoles and get:

$$\Theta_\ell(\eta_0, \vec{k}) = \int_0^{\eta_0} d\eta \; \tilde{S}(\vec{k}, \eta) \, j_\ell(k(\eta_0 - \eta)).$$

As usual, it is usefull to write $\tilde{S}(\vec{k}, \eta)$ as the product of stochastic initial condition $\mathcal{R}(\vec{k})$ and transfer function (not depending on $\hat{k}$).

Let us call this transfer function S:

$$S(\eta, k) = \frac{\tilde{S}(\eta, \vec{k})}{\mathcal{R}(\vec{k})} = \frac{(\Psi' + \Phi')e^{-\tau} + g(\Phi + \Theta_0) + \left(g\frac{\Theta_b}{k^2}\right)'}{\mathcal{R}}$$

This transfer function is called the source function of temperature anisotropies. Finally we can write:

$$\boxed{\begin{aligned} &\Theta(\eta_0, \vec{k}, \mu) = \int_0^{\eta_0} d\eta\ S(k,\eta)\, e^{-ik\mu(\eta_0-\eta)}\, \mathcal{R}(\vec{k}) \\ &\text{or} \\ &\forall \ell,\ \Theta_\ell(\eta_0, \vec{k}) = \int_0^{\eta_0} d\eta\ S(k,\eta)\, j_\ell(k(\eta_0-\eta))\, \mathcal{R}(\vec{k}). \end{aligned}}$$

We defined the transfer function $\Delta_\ell(\eta, k) = \frac{\Theta_\ell(\eta, \vec{k})}{\mathcal{R}(\vec{k})}$. So

$$\boxed{\Delta_\ell(\eta_0, k) = \int_0^{\eta_0} d\eta\ S(k,\eta)\, j_\ell(k(\eta_0-\eta))}$$

In other words, we managed to separate $\Delta_\ell(\eta_0, k)$ in a term depending on the physics: $S(k,\eta)$, and a term depending on the geometry: $j_\ell(k(\eta_0-\eta))$.

But $\overset{(\sim)}{S}\overset{(\to)}{(k},\eta)$ depends only on Ψ', Φ', Φ, Θ_0, Θ_b, not on $\Theta_{\ell \geq 1}$! So we can think of a very improved scheme for computing C_ℓ's:

Clever computation of C_ℓ's :

We could write a code doing the following:

* the code should know the system of equations in Fourier space, with Boltzmann hierarchy <u>truncated at low l</u> (typically $\ell \sim 10$) because we only need to trust result for $\Theta_0(\eta,\vec{k})$
* the code integrates this system from η_{ini} to η_0 and, <u>at each step</u>, <u>store</u> $S(k,\eta)$ in memory.
* the initial conditions should be $\forall k$, $\mathcal{R}(\vec{k})=1$, so that $\tilde{S}(\eta,\vec{k})$ in the code $= S(\eta,k)$ in the course.
* at the end, the code performs the following convolutions:

$$\begin{cases} \Theta_\ell(\eta_0,k) = \int_0^{\eta_0} d\eta\, S(k,\eta)\, j_\ell(k(\eta_0-\eta)) \\ C_\ell = \frac{1}{2\pi^2}\int \frac{dk}{k}\, |\Theta_\ell(\eta_0,k)|^2\, \mathcal{S}_{\mathcal{R}}(k) \end{cases}$$

In this approach, number of variables dramatically reduced (10 ℓ's instead of 2500 ℓ's), and time step after η_{dec} can be very large because $S(k,\eta)$ becomes a very smooth function of η after decoupling...

Limber approximation:

In the large-l limit, $j_\ell(x)$ is very peaked around $\ell+\frac{1}{2}$ and can be approximated by $\sqrt{\frac{\pi}{2\ell+1}}\,\delta(\ell+\frac{1}{2}-x)$. This will be useful in the next sections.

Instantaneous decoupling approximation:

If the optical depth is going abruptly from large values to zero at recombination, we have seen that $e^{-\tau} \simeq H(\eta-\eta_{dec})$ and $g(\eta)\simeq\delta(\eta-\eta_{dec})$. This gives

$$S(k,\eta)\simeq H(\eta-\eta_{dec})(\Phi'+\Psi')_{k,\eta}+\delta(\eta-\eta_{dec})(\Phi+\Theta_0)_{k,\eta}+\left(\delta(\eta-\eta_{dec})\frac{(\Theta_b)_{k,\eta}}{k^2}\right)'$$

and:

$$\Delta_\ell(k,\eta_0)\simeq\int_{\eta_{dec}}^{\eta_0}d\eta\,(\Phi'+\Psi')_{k,\eta}\,j_\ell(k(\eta_0-\eta))$$
$$+(\Theta_0+\Phi)_{k,\eta_{dec}}\,j_\ell(k\underbrace{(\eta_0-\eta_{dec})}_{\simeq\eta_0})+k^{-1}(\Theta_b)_{k,\eta_{dec}}\,j_\ell^{(1)}(k(\eta_0-\eta_{dec}))$$

with $j_\ell^{(1)}(x)\equiv\frac{dj_\ell}{dx}$

Since $C_\ell \sim \int\frac{dk}{k}\Delta_\ell^2\,\mathcal{S}_R$, we see that C_ℓ receives contributions mainly from $(\Theta_0+\Phi)$ and Θ_b evaluated at η_{dec} for $k\sim\frac{\eta_0-\eta_{dec}}{\ell}$ (when $j_\ell(k(\eta_0-\eta_{dec}))$ peaks), plus the integrated contribution.