## V.4) Generic contributions to the CMB spectrum

We know from the previous section that $S(k,\eta)$ receives 3 contributions: Sachs-Wolfe (SW) term in $\Theta_0 + \phi$, Doppler term in $\Theta_b/k$, and Integrated Sachs-Wolfe (ISW) in $\int d\eta\, (\phi' + \psi')$. These terms all combine with each other in the final $C_\ell$'s. The phenomenology of the ISW is different from the other ones, so we will first study this term separately.

### V.4.A) Integrated Sachs-Wolfe contribution

We focus on $S(k,\eta)^{ISW} = \left[\psi'(\vec{k},\eta) + \phi'(\vec{k},\eta)\right] e^{-\tau} \cancel{\left[\mathcal{R}(\vec{k})\right]}$

(we can assume that all perturbations are computed from IC $\boxed{\mathcal{R}(\vec{k}) = 1}$ and drop it!)

giving raise to: $\Delta_\ell^{ISW}(k,\eta_0) = \int_0^{\eta_0} d\eta\; S^{ISW}(k,\eta)\, j_\ell(k(\eta_0 - \eta))$

and to: $C_\ell^{ISW} = \frac{1}{2\pi^2} \int \frac{dk}{k} \left|\Delta_\ell(k,\eta_0)\right|^2 \mathcal{P}_{\mathcal{R}}(k)$

Instantaneous decoupling $\Longrightarrow \Delta_\ell^{ISW}(k,\eta_0) \simeq \int_{\eta_{dec}}^{\eta_0} d\eta\, (\psi' + \phi')\; j_\ell(k(\eta_0 - \eta))$

Limber approximation (not accurate, but ok for qualitative description) $\Longrightarrow$

$$\Delta_\ell^{ISW}(k,\eta_0) \simeq \begin{cases} \sqrt{\frac{\pi}{2\ell}}\, \frac{1}{k}\, (\psi' + \phi')\Big|_{k,\, \eta_0 - \frac{\ell}{k}} & \text{for } \eta_0 - \frac{\ell}{k} > \eta_{dec} \\ 0 & \text{for } \eta_0 - \frac{\ell}{k} < \eta_{dec} \end{cases}$$

So $C_\ell^{ISW} = \frac{\ell}{2\pi^2} \int_{\frac{\ell}{\eta_0-\eta_{dec}}}^{\infty} \frac{dk}{k^3} \frac{\pi}{2\ell} (\Psi'+\Phi')\Big|_{k,\eta_0-\frac{\ell}{k}} \mathcal{S}_\mathcal{R}(k)$

Things get more clear by changing variable (from $k$ to conformal time): $\eta = \eta_0 - \frac{\ell}{k}$, $\frac{dk}{k} = -\frac{d\eta}{\eta_0-\eta}$

Then:

$$C_\ell^{ISW} = \frac{\ell}{2\pi^2} \int_{\eta_{dec}}^{\eta_0} \frac{d\eta}{\eta_0-\eta} \left(\frac{\eta_0-\eta}{\ell}\right)^2 \frac{\pi}{2\ell} \left[(\Psi'+\Phi')\Big|_{\frac{\ell}{\eta_0-\eta},\eta}\right]^2 \mathcal{S}_\mathcal{R}(k)$$

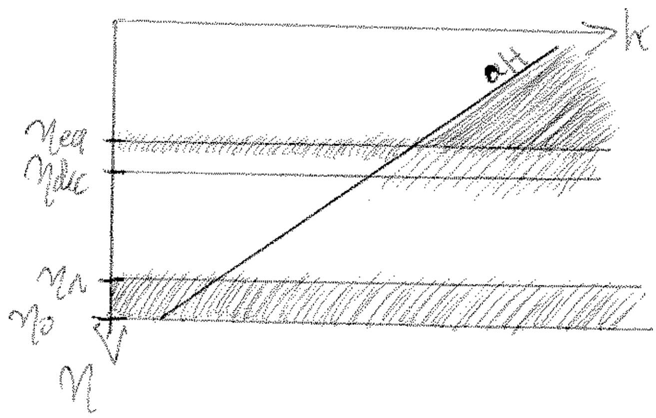
$$= \frac{1}{4\pi\ell^3} \int_{\eta_{dec}}^{\eta_0} d\eta\, (\eta_0-\eta) \left[(\Psi'+\Phi')\Big|_{\frac{\ell}{\eta_0-\eta},\eta}\right]^2 \mathcal{S}_\mathcal{R}$$

Between $\eta_{dec}$ and $\eta_0$, do we have variations of $\Psi$ and $\Phi$? We know that $\Psi$ and $\Phi$ are constant during MD, so the contribution is expected to come from $\eta_\Lambda \le \eta \le \eta_0$.

M/$\Lambda$ equality.

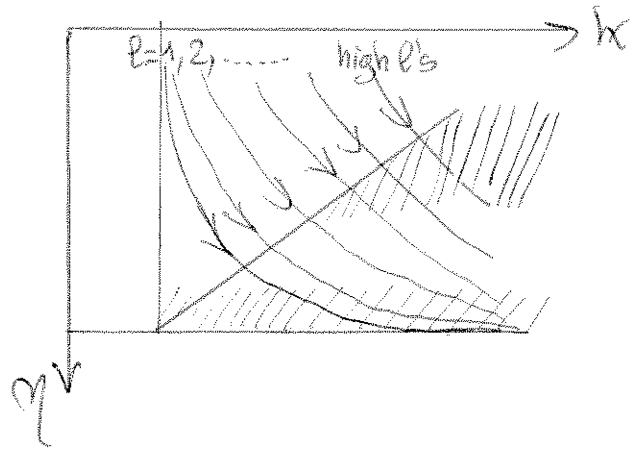
However, at $\eta = \eta_{dec}$, $\Psi$ and $\Phi$ are not fully stabilized because R/M equality took place recently, and the photon pressure cannot be completely neglected, leading to $\Phi'+\Psi' \neq 0$ on sub-horizon scales.

In $(k,\eta)$ space, regions where $(\Phi'+\Psi')$ does not vanish:




(see Chapter IV for details)

The integral must be carried along lines with $k(\eta) = \frac{\ell}{\eta_0 - \eta}$:

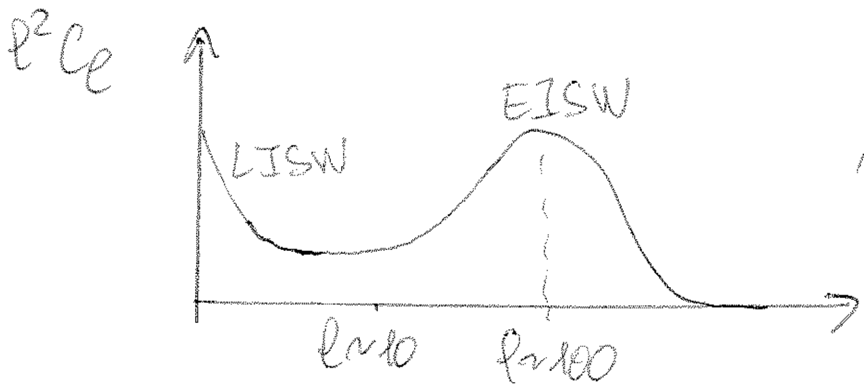


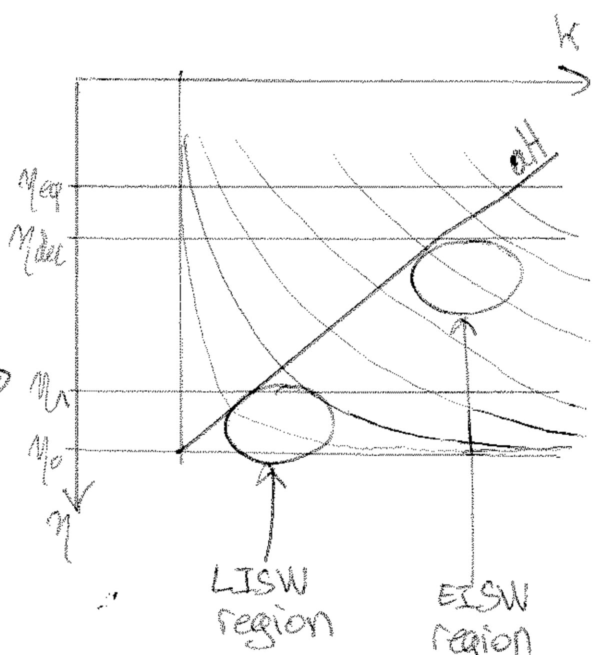

A reasoning based on the time-evolution and $k$-dependence of $\frac{\Phi+\Psi}{2}$ (constant for scales $k \ll a_{dec}H_{dec}$, decreasing with $k$ on larger $k$'s) lead to the conclusion that $C_\ell^{ISW}$ is significant for two cases:

$\left\{ \eta \geq \eta_\Lambda \, , \, k \sim \frac{1}{\eta_0} \right\}$ ("Late ISW" (LISW)) for <u>small $\ell$'s</u>

and $\left\{ \eta \sim \eta_{dec} \, , \, k \sim \frac{1}{\eta_{dec}} \right\}$ ("Early ISW" (EISW)) for $\ell \sim \frac{\eta_0 - \eta_{dec}}{\eta_{dec}} \sim 100$

Final result:



## V.4.B Sachs-Wolfe contribution for small l's:

The non-ISW piece in $\overset{\text{non-ISW}}{S}(k,\eta)$ reads $\dfrac{g(\eta)\left[\Phi+\Theta_0\right]_{\vec{k},\eta}+\left(\frac{g}{k^2}\,\Theta_b\right)_{\vec{k},\eta}}{\mathcal{R}(\vec{k})}$

Setting $\mathcal{R}(\vec{k})$ to one and using the instantaneous decoupling approximation,

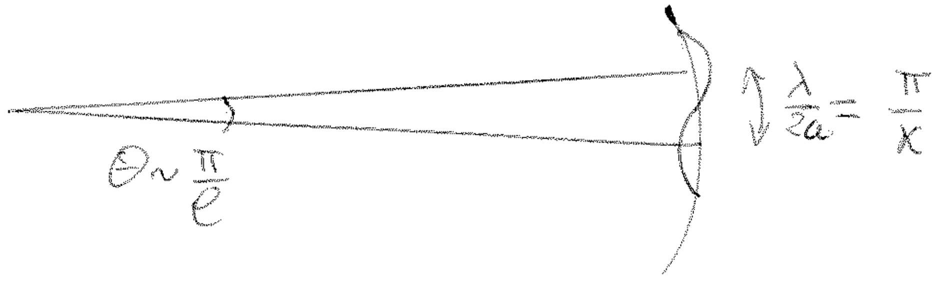
$$\overset{\text{non-ISW}}{\Delta_\ell}(k,\eta_0)\simeq\left[\Theta_0+\Phi\right]_{k,\eta_{dec}}\,j_\ell\big(k(\eta_0-\eta_{dec})\big)+k^{-1}\left[\Theta_b\right]_{k,\eta_{dec}}\,j_\ell^{(1)}\big(k(\eta_0-\eta_{dec})\big)$$

We know that dominant contribution to $j_\ell(x)$ and $j_\ell^{(1)}(x)$ comes from $x\sim\mathcal{O}(\ell)$; so, here, for fixed $\ell$, $\left|\Delta_\ell(k,\eta_0)\right|$ peaks near $k\sim\dfrac{\ell}{\eta_0-\eta_{dec}}$

So $\overset{\text{non-ISW}}{C_\ell}=\dfrac{1}{2\pi^2}\displaystyle\int_0^\infty\frac{dk}{k}\left(\overset{\text{non-ISW}}{\Delta_\ell}(k,\eta_0)\right)^2\mathcal{P}_{\mathcal{R}}(k)$ depends

mainly on $\left[\Theta_0+\Phi\right]_{\left(\frac{\ell}{\eta_0-\eta_{dec}},\,\eta_{dec}\right)}$ and $\left[\Theta_b\right]_{\left(\frac{\ell}{\eta_0-\eta_{dec}},\,\eta_{dec}\right)}$

which just reflect the relation between angles and comoving distances on the last scattering surface:




We have $\dfrac{\pi}{k}=r_{lss}\,\dfrac{\pi}{\ell}=(\eta_0-\eta_{dec})\dfrac{\pi}{\ell}\Rightarrow k=\dfrac{\ell}{\eta_0-\eta_{dec}}$

So, small $\ell$'s only receive a contribution from small $k$'s; In particular, from $k \leq a_{dec} H_{dec} = \frac{1}{\eta_{dec}}$ if $\ell \leq \frac{\eta_0 - \eta_{dec}}{\eta_{dec}} \sim 100$.

remember that comoving horizon $\sim \int \frac{dt}{a} \sim \eta$ nearly equal to comoving Hubble radius $\frac{1}{aH}$

So, for $C_\ell$'s with $\ell \leq 100$ we can use the results of Chapter IV for super-Hubble scales at the time of decoupling (during MD):

* $\Theta_0 + \phi = \frac{1}{4}\delta_\gamma + \phi = \left(-\frac{2}{5} + \frac{3}{5}\right)\mathcal{R} = \frac{1}{5}\mathcal{R}$

* $\Theta_b$ plays negligible role (note: Einstein $\Rightarrow \frac{\theta_{tot}}{k} \sim \frac{k}{aH}\phi$)

So $\Delta_\ell^{\text{non-ISW}}(k, \eta_0) \simeq \frac{1}{5} j_\ell(k(\eta_0 - \eta_{dec}))$

$\Rightarrow C_\ell^{\text{non-ISW}} = \frac{1}{2\pi^2}\int \frac{dk}{k}\frac{1}{25} j_\ell^2(k(\eta_0-\eta_{dec}))\,\mathcal{P}_\mathcal{R}(k)$

Exact solution for $\mathcal{P}_\mathcal{R}(k) = \text{cte}$ ($n_s = 1$, Harrison Zel'dovitch):

$$C_\ell^{\text{non-ISW}} = \frac{1}{2\pi^2}\frac{\mathcal{P}_\mathcal{R}}{25}\int \frac{dx}{x} j_\ell^2(x) = \frac{1}{2\pi^2}\frac{\mathcal{P}_\mathcal{R}}{25}\frac{1}{2\ell(\ell+1)}$$

So $\ell(\ell+1)\,C_\ell^{\text{non-ISW}} = \frac{\mathcal{P}_\mathcal{R}}{100\pi^2}$

If $n \neq 1$, $\ell(\ell+1)C_\ell^{\text{non-ISW}}$ gets a non-zero slope as a function of $\ell$.

## V.4.C) Sachs-Wolfe and Doppler contribution: large l's:

We need $\frac{\Theta_0+\Phi}{\mathcal{R}}$ and $\frac{\theta_b}{k\mathcal{R}}$ at the time of decoupling for $k \geq a_{dec} H_{dec}$ ($k\eta_{dec} \geq 1$), ie for modes experiencing acoustic oscillations.

- **First guess:**

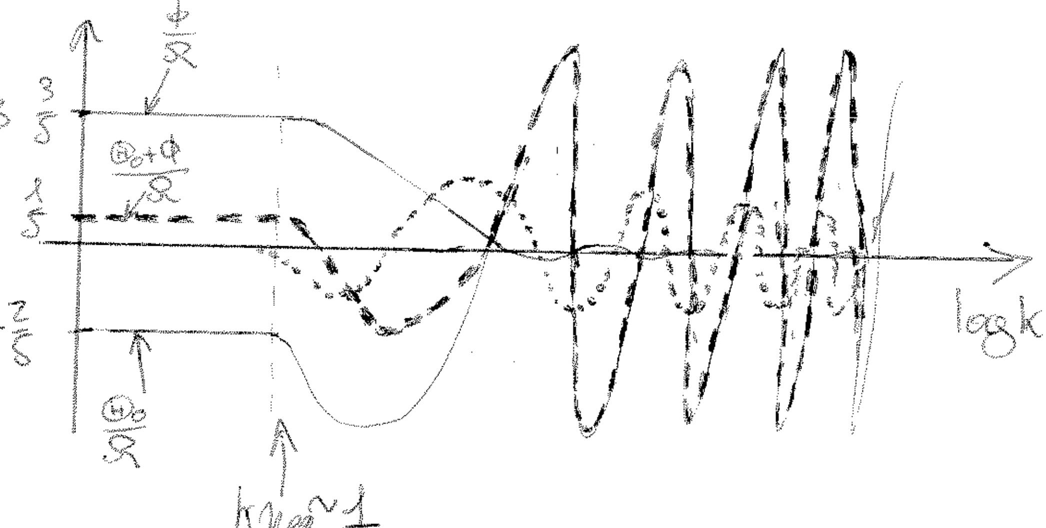
We start from a very crude approach: we take the results of Chapter IV for the sub-Hubble evolution during RD (based on a tightly coupled photon-baryon fluid with $\rho_b \ll \rho_\gamma$ and $c_s^2 = \frac{1}{3}$), and extrapolate them till decoupling! We know that at fixed $\eta$ (say, $\eta_{eq}$) we then have:

* $\frac{\Theta_0}{\mathcal{R}} = \frac{1}{4}\frac{\delta_\gamma}{\mathcal{R}}$ goes from constant on small k to $\cos(kc_s\eta)$ for $k\eta \geq 1$

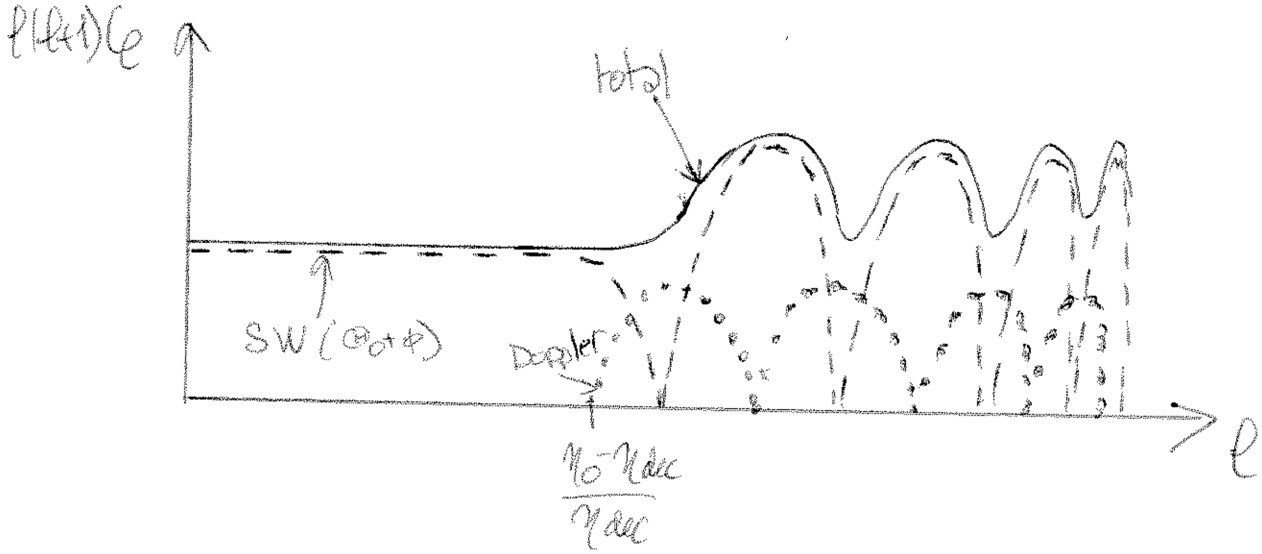
* $\frac{\Phi}{\mathcal{R}}$ goes from constant on small k to $\frac{\cos(kc_s\eta)}{(kc_s\eta)^2}$ for $k\eta \geq 1$

* until decoupling, $\theta_b = \theta_\gamma$, so $\frac{\theta_b}{k\mathcal{R}} = \frac{\theta_\gamma}{k\mathcal{R}} \left(= -3\frac{\Theta_1}{\mathcal{R}}\right)$ goes from zero to $\sin(kc_s\eta)$ (since $\theta_\gamma \propto \delta_\gamma'$) comes from continuity eq. with $|\Psi'| \ll |\delta_\gamma'|$

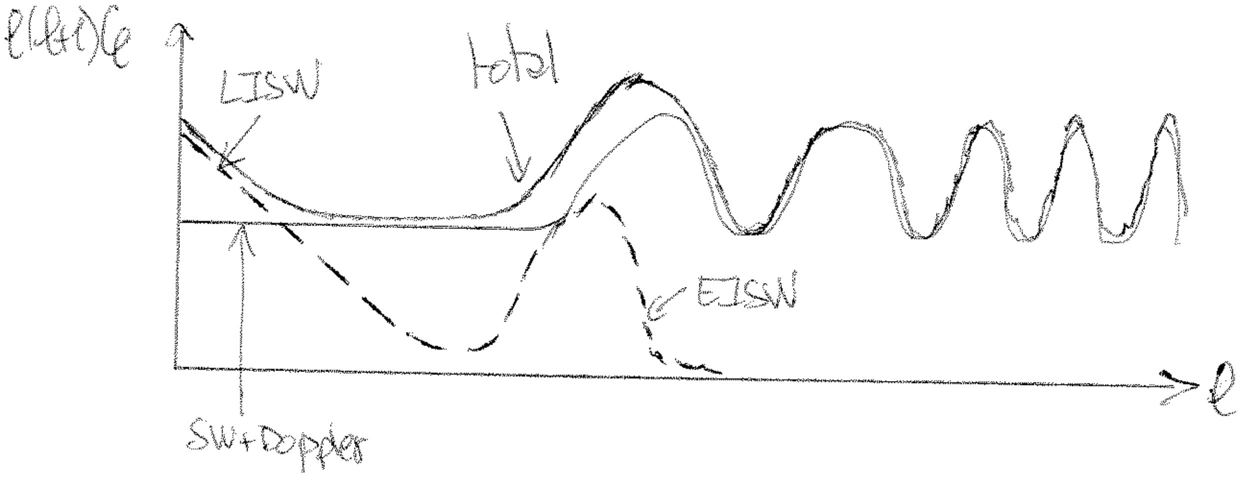
So: (extrapolating until $\eta_{dec}$!!)



The $C_\ell$'s depend on the square of this quantities (actually, not the $C_\ell$'s, but $\ell(\ell+1)C_\ell$, due to the Bessel functions); with the correspondance $k \Leftrightarrow \frac{\ell}{\eta_0-\eta_{dec}}$:




Adding ISW:




This picture is wrong for 2 reasons (mainly):

(i) we treated $\gamma$+b as fluid with $\rho_b \ll \rho_\gamma$ (leading to $\rho_{tot}\delta_{tot} = \rho_\gamma \delta_\gamma$ and $c_s^2 = \frac{1}{3}$) and self-gravitating (wrong after equality)

(ii) we treated decoupling as instantaneous

We can overcome (i) with our knowledge of exact coupled equations

## ■ photon-baryon fluid in 1st order tight coupling approximation

Boltzmann:

$$\Theta_0' + k\,\Theta_1 = \psi' \qquad (1)$$

$$\Theta_1' - \frac{k}{3}\Theta_0 + \frac{2k}{3}\Theta_2 = \frac{k}{3}\phi + \tau'\underbrace{\left(\frac{\theta_b}{3k} + \Theta_1\right)}_{=\,\frac{\theta_b-\theta_\gamma}{3k}} \qquad (2)$$

$$\Theta_2' + \cdots = \tau'\,\Theta_2 \qquad (3)$$

Euler for baryons:

$$\theta_b' = -\frac{a'}{a}\theta_b - k^2\phi + \frac{\tau'}{R}\underbrace{\left(\theta_b + 3k\,\Theta_1\right)}_{=\,\theta_b-\theta_\gamma} \qquad (4)$$

with $R \equiv \frac{3\rho_b}{4\rho_\gamma} \propto a$

$$(3) \Rightarrow \Theta_2 = \frac{1}{\tau'}\left(\Theta_2' + \cdots\right) \qquad (5)$$

$$(4) \Rightarrow \theta_b = -3k\,\Theta_1 + \frac{R}{\tau'}\left(\theta_b' + \frac{a'}{a}\theta_b + k^2\phi\right) \qquad (6)$$

* Solution at order 0 in $\tau'^{-1}$: taking $\tau'^{-1} = 0$ in (5,6) we get $\Theta_2 = 0$ and $\theta_b = -3k\,\Theta_1$.
* Solution at order 1 in $\tau'^{-1}$: insert above solutions in (5,6):

$$\begin{cases} \Theta_2 = \frac{1}{\tau'}\left(\Theta_2' + \cdots\right) \\ \theta_b = -3k\,\Theta_1 + \frac{R}{\tau'}\left(-3k\,\Theta_1' - 3k\frac{a'}{a}\Theta_1 + k^2\phi\right) \end{cases}$$

Replace in (2):

$$\Theta_1' - \frac{k}{3}\Theta_0 + \cancel{\frac{1}{\tau'}(\cdots)} = \frac{k}{3}\phi + \frac{R}{3k}\left(-3k\,\Theta_1' - 3k\frac{a'}{a}\Theta_1 + k^2\phi\right)$$

We can replace $\Theta_1$ using (1): $\Theta_1 = (\psi' - \Theta_0')/k$ ; finally:

$$\boxed{\Theta_0'' + \underbrace{\frac{a'}{a}\frac{R}{1+R}}_{\dot R/(1+R)}\Theta_0' + \underbrace{\frac{k^2}{3(1+R)}}_{k^2 c_s^2}\Theta_0 = -\frac{k^2}{3}\phi + \frac{a'}{a}\frac{R}{1+R}\psi' + \psi''}$$

Equation way more complicated than simplistic equation for $\delta_\gamma = \delta_r$ in Chapter IV. Let us stress a few important differences:

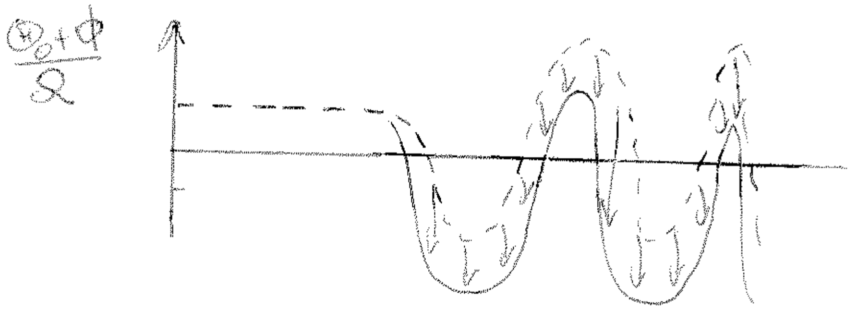
* effective mass $\frac{k^2}{3(1+R)} = k^2 c_s^2$ is not constant ($R \nearrow$, $c_s \searrow$). Oscillation depend on $\cos\left[\underbrace{\int k c_s d\eta}_{\equiv k \int \frac{dt\, c_s}{a} \equiv k\, d_s^{co}}\right]$

comoving sound horizon

* baryon damp fluctuations (friction term) when $R \gtrsim 1$, i.e. between equality and decoupling
* zero-point of oscillations, given by $\Theta_0'' = 0$ (neglecting $\Theta_0'$), depends on complicated evolution of $\phi$ and $\psi$. Assuming $\phi$ and $\psi$ to be constant at decoupling, the zero-point is given by

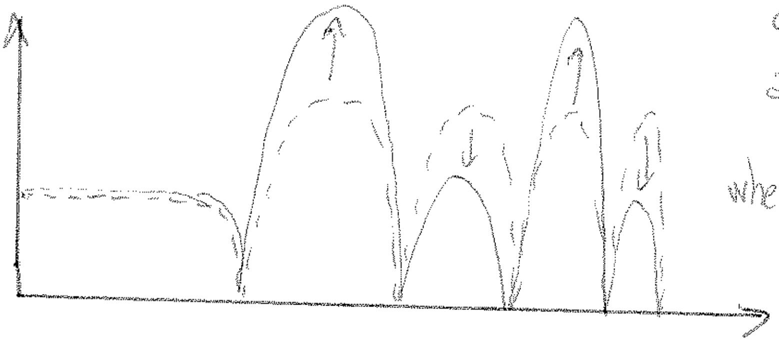
$$\frac{k^2}{3(1+R)} \Theta_0^{eq} = -\frac{k^2}{3}\phi \iff \Theta_0^{eq} = -(1+R)\phi$$

Sachs-Wolfe term $\Theta_0 + \phi$ oscillates around $-R\phi$ !!
Baryon lift zero-point from 0!




Interpretation: R affects balance between gravity and pressure in fluid. $R \nearrow$: more gravity, more compression in potential wells...

Consequence for $C_\ell$'s:




Taking into account (ii) (the fact that decoupling is not instantaneous) requires to work at higher order in $\tau'^{-1}$. In summary, Boltzmann gives:

$$\forall \ell \geq 2, \quad \Theta_\ell' = k\left(\frac{\ell}{2\ell+1}\Theta_{\ell-1} - \frac{\ell+1}{2\ell+1}\Theta_{\ell+1}\right) + \tau'\Theta_\ell$$

As long as $k < \tau'$, $\Theta_{\ell \geq 2}$ remains zero. But $\tau'$ decreases. When $k > \tau'$, $\Theta_\ell(k,\eta)$ becomes significant: transfer of power from small $\ell$'s to large $\ell$'s, starting from largest $k$. Consequence: near $\eta_{dec}$, $\frac{\Theta_0 + \Phi}{R} \searrow$ for largest $k$ (effect equivalent to multiplying by $e^{-k/k_D}$ where $k_D^{-1}$ = "damping horizon" built from $\tau'^{-1}$). This effect is called "<u>Silk damping</u>".

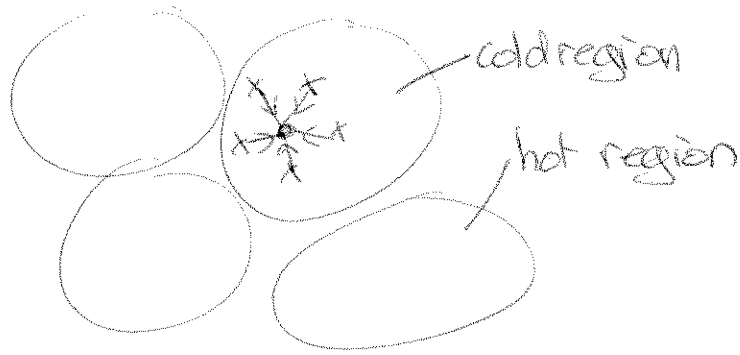
- <u>physical interpretation of Silk damping</u>

photon mean free path at time $\eta$:

$$\lambda = \text{velocity} \times [\text{scattering rate}]^{-1} \sim c/\tau'$$

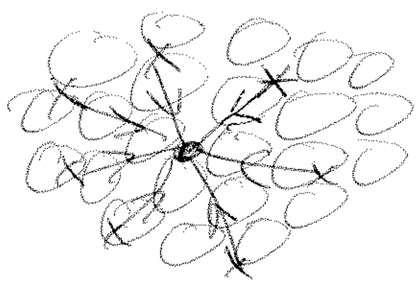
So, $\lambda$ goes <u>gradually</u> from $0$ to $\infty$ near $\eta_{dec}$.

* When $\lambda \ll$ size of $\frac{\delta T}{T}$ fluctuations, observer can only see monopole and dipole:



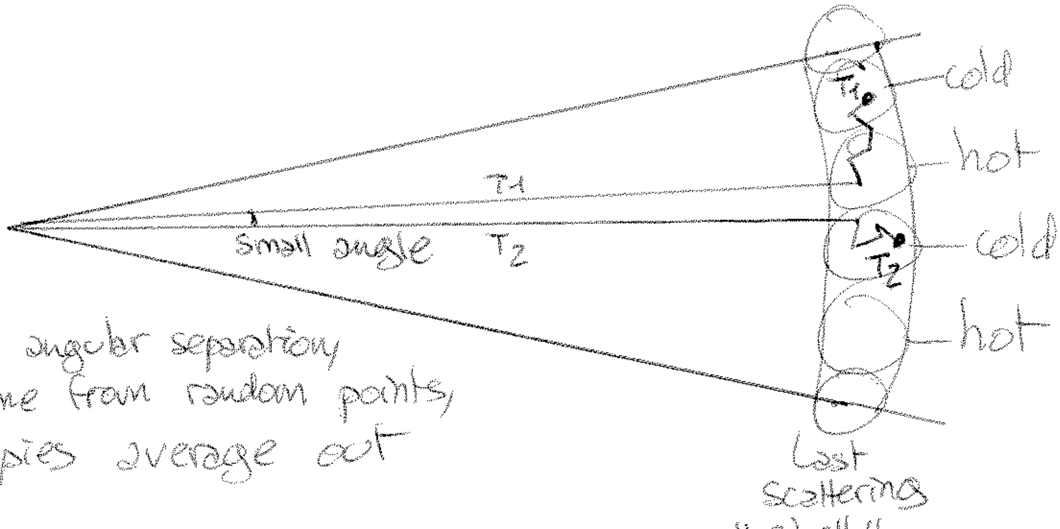

x : points where photon seen in
• <u>last scattered</u>

* when $\lambda \gtrsim$ size of $\frac{\delta T}{T}$ fluctuations, monopole averaged to zero, high $\ell$ populated:



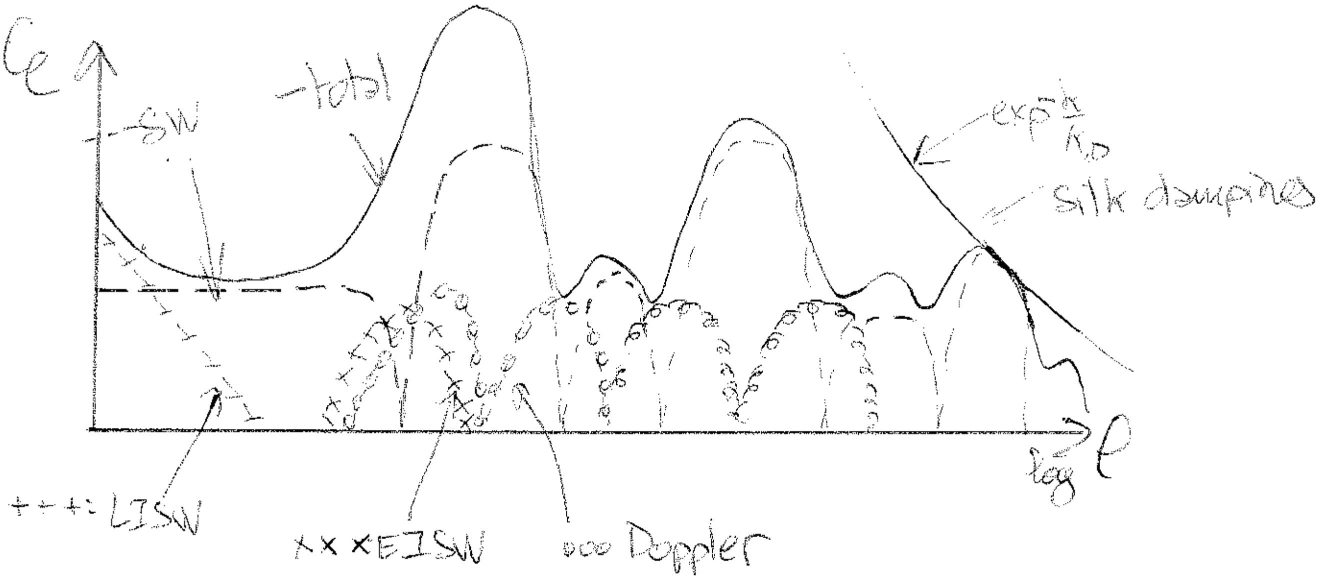
observer sees many cold / hot spots under angle depending on structure size devided by mean free path

The damping of $C_\ell$'s for large l's can be understood as a loss of coherence:




For small angular separation photons come from random points, anisotropies average out

## Summary of contributions to CMB spectrum:




where we took into account:

- → early ISW
- → late ISW
- → Sachs-Wolfe with asymetry due to baryons, and Silk damping for large $\ell$'s
- → Doppler effect